Calculus 2

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1 Integration by Parts

1.1 The Rule

Inverse of the product rule;

$(fg)' = f'g + fg'$

becomes;

$\int fg' + f'g = fg$

This format is understandable, but not very helpful. Subtracting one of the products of the integrals over to the opposite side creates a far more helpful form, typical written; $\int u\,dv = uv - \int v\,du$

1.2 Examples

1. $\int \mathbf{x\sin(x)dx}$

   $u = x; du = 1$

   $dv = sin(x); v = -cos(x)$

   $\int x sin(x)dx = x(-cos(x)) - \int (-cosx) * 1 dx$

   $= -\mathbf{xcos(x)} + \mathbf{sin(x)} + \mathbf{C}$

2. $\int \mathbf{5xe^{3x}dx}$

   $u = 5x; du = 5$

   $dv = e^{3x}; v = \frac{e^{3x}}{3}$

   $\int 5xe^{3x}dx = \frac{5xe^{3x}}{3} - 5\int \frac{e^{3x}}{3}dx$

   $= \mathbf{\frac{5xe^{3x}}{3}} - \mathbf{\frac{5}{9}e^{3x}} + \mathbf{C}$

1.3 Tips

1. Pick u such that du becomes simpler

2. Pick dv such that v doesn't get more complicated

2 Trigonometric Identities

Goal: to use trig identities to in substitution to evaluate integrals.

2.1 List of Identities

1. $sin^2(x) + cos^2(x) = 1$

2. $tan^2(x) + 1 = sec^2$

3. $cot^2(x) + 1 = csc^2$

4. $sin(2x) + 1 = 2sin(x)cos(x)$

5. $cos(2x) = 2cos^2(x) - 1 = 1 - 2sin^2(x)$

6. $sin^2(\frac{x}{2}) = \frac{1-cos(x)}{2}$

7. $cos^2(\frac{x}{2}) = \frac{1+cos(x)}{2}$

2.2 Examples

1. $\int \mathbf{sin^3(x)dx}$

   $= \int sin(x)sin^2(x)dx$

   $= \int sin(x)(1 - cos^2(x))dx$

   $[u = cos(x); du = -sin(x)dx]$

   $= -\int (1 - u^2)du$

   $= -u + \frac{u^3}{3}$

   $= \mathbf{cosx + \frac{cos^3}{3} + C}$

2. $\int \mathbf{tan^3(x)sec^4(x)dy}$

   $= \int u^3 sec^2(x)du$

   $= \int u^3(tan^2(x) + 1)du$

   $= \int u^3(u^2 + 1)du$

   $= \int u^5 + u^3 du$

   ...

3 Trigonometric Substitution

3.1 Examples

1. $\sqrt{\mathbf{1 - x^2}}\mathbf{dx}$, just the area of a circle

   $x = sin\theta, dx = cos\theta d\theta$

$\int \sqrt{1 - sin^2\theta}cos\theta d\theta$ — **trig identity**

$= \int \sqrt{cos^2\theta}cos\theta d\theta$

$= \int cos^2\theta d\theta$ — **trig identity**

$= \int \frac{1+cos(2\theta)}{2}d\theta$ — $[u = 2\theta, du = 2]$

$= \frac{\theta}{2} + \frac{sin(2\theta)}{4} + C$

$= \mathbf{\frac{sin^{-1}x}{2} + \frac{sin(2sin^{-1}x)}{4} + C}$ — **not quite done...**

3.2   Simplification

$$sin(2sin^{-1}(x)) \text{ — } [sin(2\theta) = 2sin(\theta)cos(\theta)]$$
$$= 2sin(sin^{-1}(x))cos(sin^{-1}(x))$$
$$= \mathbf{2x\sqrt{1 - x^2}}$$

3.3   Some good substitutions to remember

expression	substitution
$\sqrt{a^2 - x^2}$	$x = asin(\theta)$
$\sqrt{a^2 + x^2}$	$x = atan(\theta)$
$\sqrt{x^2 - a^2}$	$x = asec(\theta)$

4   Partial Fractions

4.1   Theorem

If $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials and $deg(P) < deg(Q)$, the f can be written as a sum of simple rational functions with either linear or quadratic denominators.

4.1.1   Example

$\frac{x^3-12x^2+15}{x^7+19x^3+12} = \frac{A}{ax+b} + \frac{B}{cx+d} + ...$
Note: $deg(P) < deg(Q)$. If not, have to divide first.

4.2   Basic Case

If f has only linear factors...
$f(x) = \frac{P(x)}{(a_1x+b_1)(a_2x+b_2)...(a_nx+b_n)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + ... + \frac{A_n}{a_nx+b_n}$
If some factors are repeated...
$f(x) = \frac{P(x)}{(a_1x+b_1)^3(a_2x+b_2)^2} = \frac{A}{a_1x+b_1} + \frac{B}{(a_1x+b_1)^2} + \frac{C}{(a_1x+b_1)^3} + \frac{D}{a_2x+b_2} + \frac{E}{(a_2x+b_2)^2}$

4.3 Irreducible Quadratic Factors

If f contains irreducible quadratic factors... (ex. $x^2 + 1$)

$f(x) = \frac{P(x)}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

Any higher-degree polynomial can be reduced down to linear and quadratic factors.

4.4 Examples

1. $\int \frac{x+3}{x^2-5x-6}dx$

   $\frac{x+3}{x^2-5x-6} = \frac{x+3}{(x-6)(x+1)} = \frac{A}{x-6} + \frac{B}{x+1}$

   $x + 3 = Ax + A + Bx - 6B$

   $x + 3 = (A+3)x + A - 6B$

   $A + B = 1, A - 6B = 3$ (SOE)

   $A = 9/7, B = -2/7$

   $\int \frac{\frac{9}{7}}{x-6} + \frac{-\frac{2}{7}}{x+1}dx$

   $= \frac{9}{2}\ln|x-6| - \frac{2}{7}\ln|x+1| + \mathbf{C}$

2. $\int \frac{x+1}{x^3-x^2}dx$ $\frac{x+1}{x^3-x^2} = \frac{x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$

   $x + 1 = Ax(x-1) + B(x-1) + Cx^2$

   At this point, we could solve for A, B, and C using a systems of equations. However, we can instead use helpful values of x and solve more easily instead.

   $x = 0 \Rightarrow 1 = A*0 + B*-1 + C*0^2 \Rightarrow \mathbf{B = -1}$

   $x = 1 \Rightarrow 2 = 0 + 0 + C \Rightarrow \mathbf{C = 2}$

   $x = 2 \Rightarrow 3 = 2A + (-1) + 8 \Rightarrow \mathbf{A = -2}$

   Using this, we can now rewrite our original integral in a much nicer form.

   $\int \frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{x+1}dx$

   $= -\mathbf{2}\ln \mathbf{x} + \frac{\mathbf{1}}{\mathbf{x}} + \mathbf{2}\ln \mathbf{x} + \mathbf{1} + \mathbf{C}$

3. $\int \frac{10}{(x-1)(x^2+9)}dx$

   $\frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9}$

   $10 = A(x^2 + 9) + (Bx + C)(x - 1)$

   We can use the same idea as above; rather than solving using SOE, we can instead simply pick nice values for x

   $x = 1 \Rightarrow 10 = A(10) \Rightarrow \mathbf{A = 1}$

   $x = 0 \Rightarrow 10 = 9A + C(-1) \Rightarrow \mathbf{C = -1}$

   $x = 2 \Rightarrow \mathbf{B = -1}$

Rewrite integral as...

$\int \frac{1}{x-1} + \frac{-x-1}{x^2+9} dx$

$= \int \frac{1}{x-1} + \frac{-x}{x^2+9} - \frac{1}{x^2+9} dx$

First factor uses basic integration, second uses u substitution, and the last uses the fact that $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} tan^{-1}(\frac{x}{a})$

$= \ln|\mathbf{x-1}| - \frac{1}{2}\ln|\mathbf{x^2+9}| - \frac{1}{3}\mathbf{tan^{-1}}(\frac{\mathbf{x}}{\mathbf{3}}) + \mathbf{C}$

4. $\int \frac{x^3+4}{x^2+4} dx$

   Since $deg(P) > deg(Q)$, we have to do some work first by dividing P by Q.

   $(x^3+4)/(x^2+4) = x + \frac{-4x+4}{x^2+4}$

   $\int x + \frac{-4x+4}{x^2+4} dx$

   This way, we already have a irreducible quadratic factor, so we can integrate normally.

   $= \frac{\mathbf{x^2}}{\mathbf{2}} - \mathbf{2}\ln|\mathbf{x^2+4}| + \mathbf{2tan^{-1}}(\frac{\mathbf{x}}{\mathbf{2}}) + \mathbf{C}$

5. $\int \frac{dx}{2\sqrt{x+3}+x}$

   In this example, we don't even have a rational function. We need to transform it first through substitution.

   $u = \sqrt{x+3}$

   $x = u^2 - 3, 2udu = dx$

   $\int \frac{2udu}{(u+3)(u-1)}$

   Work as before...

   $= \frac{3}{2}\ln(\sqrt{x+3}+3) + \frac{1}{2}\ln(\sqrt{x+3}-1) + C$

5 Integration Strategies

How do I know what technique to use?

5.1 Simplifying

A helpful way to think about all this is to simplify: in reality, we only have 2 integration techniques, with other techniques just subsets/algebraic manipulation.

1. Substitution

2. Integration by Parts

 So which one to use?

1. **Try u-substitution first (with simplification). Did it work?**

2. **If not, try integration by parts (again with simplification). Did it work?**

3. **If not, think of other algebraic tricks to manipulate the problem**

    (a) rational function $\rightarrow$ partial fractions

    (b) trig $\rightarrow$ trig identities

    (c) $\sqrt{a^2 \pm x^2}$-type $\rightarrow$ trig subs, trig identities

6 Integration Tables

Tables found in most calculus textbooks, of many common integrals with the work shown. Essentially obsolete nowadays.

7 Approximate Integration

7.1 An Example Demonstrating Need

ex) $\int_0^5 e^{x^2} dx$
    $= F(5) - F(0)$ (normally, with FTC)

However, e^{x^2} does not have an anti-derivative that we can write with elementary functions.

Instead, we have to approximate the value of this (using, say, Riemann sums).

7.2 Riemann Approximation

Approximate $\int_0^4 \sqrt{x} dx$ using right-hand rule w/ intervals.
    $\Delta x(\sqrt{1} + \sqrt{2} + \sqrt{3} + \sqrt{4}) \approx$ **6.146**

7.3 Midpoint and Trapezoidal Rules

$\int_0^4 \sqrt{x} dx$

7.3.1 Midpoint

Same as Riemann, but using the midpoint of each rectangle approximation.
    $\Delta x(\sqrt{\frac{1}{2}} + \sqrt{\frac{3}{2}} + \sqrt{\frac{5}{2}} + \sqrt{\frac{7}{2}}) \approx$ **5.384**

7.3.2 Trapezoidal

Same as Riemann, but adjusting the "top" side to form more approximate trapezoids.
    $\Delta x(\frac{\sqrt{0} + 2\sqrt{1} + 2\sqrt{3} + \sqrt{4}}{2}) \approx$ **5.146**

7.3.3 General Formulas

1. **Midpoint Rule**

   $\int_a^b f(x)\,dx \approx \sum_{i=0}^{n-1} f(\frac{x_i + x_{i+1}}{2})\Delta x$

2. **Trapezoidal Rule**

   $\int_a^b f(x)\,dx \approx \frac{1}{2}\Delta x f(x_0) + 2f(x_1) + ... + 2f(x_{n-1}) + f(x_n)$ where $\Delta x = \frac{b-a}{n}$

7.4 Simpson's Rule

1. Divide the function into an even number of points

2. Use every 3 points to create a number of parabolas, imitating the curve

7.4.1 Formula

$\int_a^b f(x)\,dx \approx \frac{1}{3}\Delta x(f(x_0) + 4f(x_1) + 2f(x_2) + ... + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n))$
where $\Delta x = \frac{b-a}{n}$ and n is an even number.

   ex) $\int_0^4 \sqrt{x}\,dx = \frac{1}{3}(1)(\sqrt{0} + 4\sqrt{1} + 2\sqrt{2} + 4\sqrt{3} + \sqrt{4} \approx 5.25)$

7.5 Error Bounding

7.5.1 What makes an approximation bad?

In general, our rules for approximation assume a nice, smooth, not-too-crazy curve. Very curvy (second and higher derivatives are large) functions cause worse approximations.

   Things that matter in approximations:

1. Higher order derivatives

2. The number of intervals

3. Length of interval of integration $(b - a)$

7.5.2 Error Bounds for Trapezoid and Midpoint Approximation

If E_T and E_M are the errors in the trapezoidal and midpoint rules, respectively, and if $f''(x) \leq K$ for all x on $[a, b]$, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \text{ and } |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

   where n is the number of intervals of integration.
ex) Suppose we approximate $\int_2^6 e^{-2x}\,dx$ using the midpoint with $n = 5$. What is the largest the error could be?
   $f'(x) = -2e^{-2x}, f''(x) = 4e^{-2x}$
   $|f''(x)| \leq 4e^{-4} = \frac{4}{e^4}$
   $|E_M| \leq \frac{\frac{4}{e^4}(6-2)^3}{24(5)^2}$

7.5.3   Error Bounds for Simpson's Rule

If E_S is the error in Simpson's Rule and if $f^{(4)}(x) \leq K$ for all x on $[a, b]$, then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

ex) Suppose we want to approximate $\int_0^1 sin(2x)\, dx$ to within a maximum error of .000002. How large do we need to let n be? (Using Simpson's).

$f'(x) = 2cos(2x)$
$f''(x) = -4sin(2x)$
$f^{(3)}(x) = -8cos(2x)$
$f^{(4)}(x) = 16sin(2x)$
Find n...
$0.000002 \leq \frac{16(1)^5}{180n^4}$
$n > 14.5$

We need an even number of intervals for Simpson's, therefore n must be at least 16.

7.6   Improper Integrals

7.6.1   A Use Case

e) What is the area between the graph of $\frac{1}{x^2}$ and the x-axis?

Can't just use a standard integral: the graph of $\frac{1}{x^2}$ continues forever to the right. Therefore, we have to deal with infinity in some way... Need to use a limit!

7.6.2   Definition

If $\int_a^b f(x)\, dx$ exists for all b, we can define

$$\int_a^\infty f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx.$$

If the limit does not exist, we say the improper integral is *divergent.* If the limit does exist, it is *convergent.*

Similarly, we define

$$\int_{-\infty}^b f(x)\, dx = \lim_{a \to -\infty} \int_a^b f(x)\, dx.$$

7.6.3   Examples

ex) What is the area between the graph fo $\frac{1}{x}$, the line $x = 1$, and the x-axis.

$$\int_1^\infty \frac{1}{x}\, dx = \lim_{b \to \infty} \int_1^b \frac{1}{x}\, dx$$
$$= \lim_{b \to \infty} \ln |x|]1, b$$
$$= \lim_{b \to \infty} \ln b - \ln 1$$
$$= \infty$$

Therefore, this interval is divergent.
For negative to positive infinity intervals...

$$\int_{-\infty}^{\infty} f(x)\,dx =$$
$$\int_{a}^{\infty} f(x)\,dx + \int_{-\infty}^{a} f(x)\,dx$$

for any value a.

ex) $\int_{-\infty} \infty \frac{x^2}{9+x^6}\,dx$

$$\lim a \to -\infty \int_{a}^{0} \frac{x^2}{9+x^6}\,dx + \lim_{b \to \infty} \int_{0}^{b} \frac{x^2}{9+x^6}\,dx$$
$$= \lim_{a \to -\infty} \tfrac{1}{9} tan^{-1}(\tfrac{x^3}{3})|[a,0] + \lim_{b \to \infty} \tfrac{1}{9} tan^{-1}(\tfrac{x^3}{3})|[0,b]$$
$$= 0 - \tfrac{1}{9}(\tfrac{-\pi}{2}) + \tfrac{1}{9}(\tfrac{\pi}{2}) - 0$$
$$= \mathbf{\tfrac{pi}{9}}$$

Non-infinite answer, therefore, the interval is convergent.

7.6.4   More Impropriety

ex) $\int_{0}^{4} \frac{1}{x-1}\,dx$
    This is improper because it has an asymptote at $x = 1$.
$\int_{0}^{1} \frac{1}{x-1}\,dx + \int_{1}^{4} \frac{1}{x-1}\,dx$
$= \lim_{b \to 1-} \int_{0}^{b} \frac{1}{x-1}\,dx + \lim_{a \to 1+} \int_{a}^{4} \frac{1}{x-1}\,dx$
$= \lim_{b \to 1-} \ln|x-1||[0,b]...$
$= -\infty$
Therefore, the integral is divergent.

7.6.5   A Comparison Theorem

Can be used to test for convergence or divergence of an improper integral; cannot find the value of the integral.

$$\text{If } 0 \le g(x) \le f(x) \text{ for all } x \ge a,$$
$$\text{Then}$$
$$\int_{a}^{\infty} f(x)\,dx \text{ converges} \to \int_{a}^{\infty} g(x)\,dx \text{ is convergent.}$$
$$\int_{a}^{\infty} g(x)\,dx \text{ diverges} \to \int_{a}^{\infty} f(x)\,dx \text{ is divergent.}$$

ex) Show that $\int_{1}^{\infty} e^{-x^2}\,dx$ is convergent.
If $x \ge 1$, then $e^{-x^2} \le e^{-x}$
$\int_{1}^{\infty} e^{-x}\,dx = \lim_{b \to \infty} -e^{-x}|[1,b]$
$= \lim_{b \to \infty} -e^{-b} + e^{-1}$
$= \tfrac{1}{e}$
This is convergent, and less than the original, therefore the original is also convergent.

8   Arc Length

8.1   Formula

$\int_a^b \sqrt{1 + (f'(x))^2}\, dx$

8.2   Examples

1. What is the distance traveled by a particle traveling along the $x^3 = y^2$ from $(0,0)$ to $(4,8)$.

   $y = x^{3/2}$

   $y' = \frac{3}{2} x^{\frac{1}{2}}$

   Arc Length $= \int_0^4 \sqrt{1 + (\frac{3}{2}\sqrt{x})^2}\, dx$

   $... = \frac{8}{27}(10^{\frac{3}{2}} - 1)$

2. What is the arc length of the curve $y = x^{\frac{1}{3}}$ from -1 to 1?

   $\int_{-1}^1 \sqrt{1 + (\frac{1}{3x^{\frac{2}{3}}})^2}\, dx$

   This is an improper integral! We could continue integrating this, but instead we could integrate with respect to y to create a nicer problem.

   $x = y^3$

   $\int_{-1}^1 \sqrt{1 + (3y^2)^2}\, dy$

8.3   Arc Length Function

If $y = f(x)$, $S(x) = \int_a^x \sqrt{1 + (f'(t))^2}\, dt$ is the arc length from a to x.

$\frac{ds}{dx} = \sqrt{1 + (\frac{dy}{dx})^2}$

$ds = dx\sqrt{1 + \frac{dy^2}{dx^2}}$

$ds^2 = dx^2 + dy^2$

This is really just the Pythagorean theorem!

9   (Lateral) Area of a Surface of Revolution

9.1   Surface Area of a Frustum

How do we calculate the area of this not-quite-a-cylinder?

We could compute it as a difference of 2 cones.

From here, we can derive the formula $SA = 2\pi r l$

9.2 Area of a Surface of Revolution

$$SA = \sum 2\pi r l$$
$$= \sum 2\pi (\tfrac{f(x_i)+f(x_{i+1})}{2})\sqrt{1 + f'(c_i)^2}\Delta x$$
$$\text{As } \Delta x \to 0 f(x_{i+1} \approx f(c_i), \text{ so}$$
$$= \sum 2\pi f(c_i)\sqrt{1 + (f'(c))^2}\Delta x$$
$$\text{As } n \to \infty, \Delta x \to 0$$
$$= 2\pi \int f(x)\sqrt{1 + (f'(x))^2}\,dx$$

This is our surface area.

9.3 Example

1. What is the surface area of a sphere of radius r?

$$y = \sqrt{r^2 - x^2}$$
$$SA = 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{1 + (\tfrac{-2x}{2\sqrt{r^2 - x^2}})}\,dx$$
$$= 2\pi \int_{-r}^{r} \sqrt{r^2 - x^2}\sqrt{\tfrac{r^2}{r^2 - x^2}}\,dx$$
$$= 2\pi \int_{-r}^{r} r\,dx$$
$$= 4\pi r^2$$

This matches our known formula

2. What is the area of the surface of revolution obtained by rotating the curve $y = x^{1/3}$, from $1 \le y \le 2$, around the y-axis?

$$SA = 2\pi \int_{1}^{2} x\sqrt{1 + (\tfrac{dx}{dy})^2}\,dy$$
$$= 2\pi \int_{1}^{2} y^3 \sqrt{1 + (3y^2)^2}\,dy$$
$$= \pi/27(145\sqrt{145} - 10\sqrt{10})$$

10 Sequence

10.1 Definition

An infinite list of numbers, in order, indexed by natural numbers.

10.2 Expression Sequences

Common notation: $\{a_1, a_2, a_3, ...\} = \{a_n\} = \{a_n\}_{n=1}^{\infty}$
  ex) $\{3/n\} = \{3/1, 3/2, 3/3, 3/4...\}$
  ex) $\{0, 3, 8, 15, 24, ...\} = \{a_n\}$ where $a_n = n^2 - 1$

Not all sequences can be expressed via formula, however, ex) $\{a_b\}$ where $a_n = nth$ digit of π, or $\{F_n\}$ where $F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ (Fibonacci sequence).

These are a recursive formula. These can sometimes, but not always, turned into an explicit formula.

10.3   Sequence Terminology

1. A sequence is *increasing* if $a_n < a_{n+1}$ for all $n \geq 1$.

2. A sequence is *decreasing* if $a_n > a_{n+1}$ for all $n \geq 1$.

3. A sequence is *monotonic* if it is increasing or decreasing.

4. A sequence is *bounded above* if there exists some M such that $a_n \leq M$ for all $n \geq 1$.

5. A sequence is *bounded below* if there exists some M such that $a_n \geq M$ for all $n \geq 1$.

6. A sequence is *bounded* if it is bounded from above and below

10.4   Limits of a Sequence

10.4.1   Definition

$\lim_{n \to \infty} a_n = L$ if for all $\epsilon > 0$ there is some N such that if $n < N$, then $|a_n - L| < \epsilon$.

If this limit exists (ie, closes to a point), then the function converges. Otherwise, the function diverges (ie, fluctuates between two points, continues to infinity).

10.4.2   A Theorem

If a_n is given by a function f $(a_n = f(n))$, and $\lim_{x \to \infty} f(x) = L$, then $\lim_{n \to \infty} a_n = L$.

So all the limit laws we have for functions will work for sequences as well.

10.4.3   Examples

ex) Does $\{\frac{(-1)^n}{\sqrt{n}}\}$ converge?

$\quad f(x) = \frac{(-1)^x}{\sqrt{x}}$

$\quad$ Apply Squeeze Theorem: $\frac{-1}{s\sqrt{n}} < \frac{(-1)^x}{\sqrt{x}} < \frac{1}{s\sqrt{x}}$

$\quad \lim_{x \to \infty} \frac{1}{2\sqrt{x}} = 0$

$\quad \lim_{x \to \infty} -\frac{1}{2\sqrt{x}} = 0$

$\quad$ Therefore, $\lim_{x \to \infty} \frac{(-1)^x}{2\sqrt{x}} = 0$

$\quad$ ex) Does the sequence $\{a_n\}$ where $a_1 = 3$ and $a_{n+1} = \frac{a_n}{n}$ converge?

Each term multiplies the previous by $1/n$.

$a_n = \frac{3}{1*2*3*...(n-1)} = \frac{3}{(n+1)!}$,

$\lim_{n\to\infty} a_n = 0$

ex) For what values of r does $\{r^n\}$ converge?

$f(x) = r^x$

If $r > 1$, exponential growth. If $0 < r < 1$, exponential decay. In exponential decay, $f(x)$ is heading towards a limit, and therefore it converges.

But what if $r < 0$?

If $-1 < r < 0$, then by Squeeze Theorem, converges to 0.

If $r <= -1$, diverges.

In summary, converges if $-1 < r <= 1$. If $01 < r < 1$, converges to 0. If $r = 1$, converges to 1.

10.4.4 Another Theorem

Every bounded, monotonic sequence is convergent.

Can't go in one direction infinitely since it is bounded; similarly, can't go another direction since it is monotonic.

11 Series

11.1 Zeno's Paradox

Someone is walking towards a wall. He, of course, at some point walks halfway, leaving half the total distance remaining to walk. Then, he walks half of the remaining, then again and again, etc., always having a little left to reach the wall.

So what's going on? Does $1/2 + 1/4 + 1/8 + ...$ equal 1?

11.2 Decimal Representations

$\pi \approx 3.14159 = 3 + 1x10^{-1} + 4x10^{-1} + 1x10^{-3} + 5x10^{-4} + ...$

These kinds of irrational decimals are in fact infinite sums.

11.3 Power Series

Claim: $\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$

$1 = (1 + x + x^2 + ...)(1 - x)$

$1 = 1(1-x) + x(1-x) + x^2(1-x) + ...$

$1 = 1 - x + x - x^2 + x^2 - x^3 + x^3 + ...$

$1 = 1$

This like a true statement, and demonstrates a way to represent a fraction as an infinite series.

11.4   Issues

$0 = 0 + 0 + 0 + 0 + ...$
$\quad = (1 - 1) + (1 - 1) + (1 - 1) + ...$
$\quad = 1 + (-1 + 1) + (-1 + 1) + (-1 + 1) + ...$
$\quad = 1 + 0...$
$\quad = 1$

Right? Well, we can't just use series like this all the time; there are some caveats.

11.5   Defining Infinite Series

$\sum_{i=1}^{\infty} a_i = \lim_{n \to \infty} \sum_{i=1}^{n} a_i$

The last summation is often written as s_n.

If the limit does not exist, then the sum is divergent.

11.5.1   Example

ex) $\sum_{n=1}^{\infty} 3n + 2$
$\quad = (3 + 2) + (6 + 2) + (9 + 2) + (12 + 2) + ...$
$5 + 8 + 11 + 14 + ...$
Partial sums: $5, 13, 24, 38, ...$
This is divergent: do not approach any finite number.
ex) $\sum_{n=1}^{\infty} (\frac{1}{2})^n$
$= 1/2 + 1/4 + 1/8 + 1/16 + ...$
$s_1 = 1/2$
$s_2 = 3/4$
$s_3 = 7/8$
...
It appears that $s_n = \frac{2^n - 1}{2^n}$, meaning that $\lim_{n \to \infty} \frac{2^n - 1}{2^n} = \lim_{n \to \infty} 1 - \frac{1}{2^n} = 1$.
ex) This is the harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n}$
$= 1 + 1/2 + 1/3 + 1/4 + 1/5$
Converge or diverge?
This actually diverges; although the partial sums are getting smaller, there is no real pattern, and this limit equals ∞.

11.6   Theorems

1. If $\sum a_n$ converges, then $\lim_{n \to \infty} a_n = 0$.

   This is often used in the divergence test: If $\lim_{n \to \infty} \neq 0$, then $\sum a_n$ diverges.

   Note: Converse of this statement is not true; $\lim_{n \to \infty} a_n = 0$ does not necessarily mean that the sum converges; the harmonic series falls into this category.

2. Since series are defined via limits, the usual limit laws hold.

   If $\sum a_n$ and $\sum b_n$ both converge, then

   (a) $\sum c a_n = c \sum a_n$
   (b) $\sum (a_n + b_n) = \sum a_n + \sum b_n$
   (c) $\sum (a_n - b_n) = \sum a_n - \sum b_n$

   However, if either sum is divergent, then these rules will not always hold.

11.7   General Examples

1. Write down the first four partial sums of $\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n}$.

   $= -1/2 + 2/4 + -3/8 + 4/16 + ...$

   $s_1 = -1/2$

   $s_2 = 0$

   $s_3 = -3/8$

   $s_4 = -1/8$

2. What is $\sum_{n=1}^{\infty} r^{n-1}$?

   This is an example of a geometric series.

   $s_n = 1 + r + ... + r^{n-1}$

   $r s_n = r + r^2 + ... + r^n$

   $s_n - r s_n = 1 - r^n$

   $s_n = \frac{1-r^n}{1-r}$

   $\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{1-r^n}{1-r} = \frac{1}{1-r}$ if $|r| < 1$. Otherwise, diverges.

3. What is $\sum_{n=1}^{\infty} \frac{(-4)^{n+1}}{4^n}$?

   $= 9/4 + -27/16 + ...$

   Rewrite:

   $= \sum \frac{(-3)^n (-3^n - 1)}{4(4)^{n-1}}$

   $= \frac{9}{4} \sum_{n=1}^{\infty} \left(-\frac{3}{4}\right)^{n-1}$

   This is a geometric series.

   $= \frac{9}{4} \left(\frac{1}{1-\left(-\frac{3}{4}\right)} \right)$

   $= \frac{9}{4} * \frac{1}{\frac{7}{4}}$

   $= \frac{9}{7}$

4. Write 0.888... as a ratio of two integers.

$= 8/10 + 8/100 + 8/1000 + ...$

$= 8x10^{-1} + 8x10^{-2} + 8x10^{-3} + ...$

$= \sum_{n=1}^{\infty} 8(\frac{1}{10})^n$

$= \sum_{n=1}^{\infty} \frac{8}{10}(\frac{1}{10})^{n-1}$

$= \frac{8}{10} * \frac{1}{1-\frac{1}{10}}$

$= \frac{8}{9}$

5. True or false: if two series, $\sum a_n$ and $\sum b_n$ both diverge, then $\sum(a_n + b_n)$ also diverges.

False; take:

$\sum a_n = \sum 1, \sum b_n + \sum -1$

$\sum(1 + -1)$

$= 0$

6. True or false: if $\sum a_n$ diverges and $\sum b_n$ converges, then $\sum(a_n + b_n)$ diverges.

True; $\sum b_n$ is a single number. $suma_n$ either goes to infinity, or something similar. Therefore, adding the two together still results in some sort of "infinity".

A more rigorous proof by contradiction:

Suppose it was false; $\sum a_n$ diverges, $\sum b_n$ converges, and $\sum(a_n + b_n)$ converges.

Then: $\sum(a_n + b_n) - \sum(b_n)$

$= \sum(a_b + b_n - b_n)$

$= \sum a_n$, which should converge, but doesn't...

12   The Integral Test

12.1   Why

Sometimes it is very hard to compute the limit of a series.

Can we at least tell if it converges or diverges?

12.2   Picturing Partial Sums

Suppose $a_n = f(n)$ for some function f.

$\sum a_n = 2 + 3 + 2 + 1 + 1/2$

This can be drawn as a number of rectangles of different heights of width 1. The sum is just the sum of the areas; very similar to Riemann sums.

12.2.1 Examples

1. $a_n = \frac{1}{n^2}$

   $\sum a_n = 1 + 1/4 + 1/9 + 1/16 + ...$

   $\sum a_n \leq \int_1^\infty \frac{1}{x^2}\, dx + 1$

2. $a_n = \frac{1}{\sqrt{n}}$

   $\sum a_n \geq \int_1^\infty \frac{1}{\sqrt{x}}\, dx$

12.3 The Test

Suppose $a_n = f(n)$, where f is a continuous, positive, decreasing function on $[1, \infty)$. Then the series $\sum a_n$ is convergent if and only if the integral $\int_1^\infty f(x)\, dx$ is convergent.

12.4 Remainders

$\sum a_n \lim_{n \to \infty} s_n$, where $s_n = a_1 + a_2 + ...$

Compare s_n with $\sum a_n$

$\int_{n+1}^\infty f(x)dx \leq R_n \leq \int_n^\infty f(x)dx$

The error between the nth partial sum s_n and s can be bounded by the above.

$s_n + R_n = s$

$s_n + \int_{n+1}^\infty f(x)dx \leq s \leq s_n + \int_n^\infty f(x)dx$

12.5 Examples of the Integral Test

1. $\sum_{n=1}^\infty \frac{n}{n^2+1}$

   $\int_1^\infty \frac{x}{x^2+1}dx = \lim_{b \to \infty} \int_1^b \frac{x}{x^2+1}dx = \lim_{b \to \infty} 1/2 \ln x^2 + 1|_1^b$

   $= \lim_{b \to \infty} 1/2 \ln b^2 + 1 - 1/2 \ln 2$

   $= \infty$

   The series diverges.

2. $\sum_{n=1}^\infty \frac{1}{n^3}$

   $\int_1^\infty \frac{1}{x^3}dx = \lim_{b \to \infty} -\frac{1}{2x^2}|_1^b$

   $= \lim_{b \to \infty} -\frac{1}{2b^2} + 1/2$

   $= 1/2$

   Therefore, the series converges (though not to 1/2...)

   Furthermore...

   $\int_{n+1}^\infty \frac{1}{x^3}dx \leq R_n \leq \int_n^\infty \frac{1}{x^3}dx$

   $\frac{1}{2(n+1)^2} \leq R_n \leq \frac{1}{2n^2}$

   If we wanted to be within a, we need $R_n \leq a$. If $\frac{1}{2n^2} \leq a$, this works.

3. **p-series**

   Series of form $\sum \frac{1}{n^p}$.

   For what values of p does a p-series converge?

   If $p \leq 0$, clearly diverges.

   Otherwise, $\int_1^\infty \frac{1}{x^p} = \lim_{b \to \infty} -\frac{1}{px^{p-1}}\big|_1^b$

   $= \lim_{b \to \infty} -\frac{1}{pb^{p-1}} + 1/p$

   This converges if $p > 1$.

   Note: if $p = 1$, it is the harmonic series.

   p-series converges if $p > 1$.

13   The Comparison Test

1. If $b_n \geq a_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.

2. If $b_n \leq a_n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

13.1   Examples

1. $\sum_5^\infty \frac{3}{n-4}$

   $= 3 \sum \frac{1}{n-4}$

   Very close to $\sum 1/n$; we can compare to it.

   $\frac{1}{n-4} \geq \frac{1}{n}$

   Since $\sum 1/n$ diverges, so does our sum.

2. $\sum_{n=3}^\infty \frac{n-1}{(n+1)^3}$

   $\frac{n-1}{(n+1)^3} \leq \frac{n+1}{(n+1)^3} = \frac{1}{(n+1)^2} \leq \frac{1}{n^2}$

   $\sum \frac{1}{n^2}$ converges, therefore our series converges as well.

3. $\sum \frac{\sqrt{n^4-1}}{n^3+n}$

   This is kind of like $1/n$; $\sqrt{n^4 - 1} \approx \sqrt{n^4} = n^2$ and $n^3 + n \approx n^3$; $\approx \frac{n^2}{n^3}$

   This is hard to prove algebraically however; this is where we use Limit Comparison.

13.2   Limit Comparison

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If $\lim_{n \to \infty} \frac{a^n}{b^n} = c$, where $0 < c < \infty$, then either both series converge or both series diverge.

   Ex) $\sum \frac{\sqrt{n^4-1}}{n^3+n}$

   $(\lim_{n \to \infty} \frac{\sqrt{n^4-1}}{n^3+n})/\frac{1}{n} = \lim_{n \to \infty} \frac{\sqrt{n^4+1}/\sqrt{n^4}}{1+\frac{1}{n^2}} = 1$

   Since this series diverges, then so does our original series.

14 Alternating Series

A series whose terms alternate between positive and negative.

$\sum a_n = \sum (-1)^{n-1} b_n \, or \, \sum (-1)^n b_n, \; b_n \geq 0$

14.1 Alternating Series Test

If the series $\sum_{n=1}^{\infty} (-1)^n b_n$ satisfies:

1. $b_{n+1} \leq b_n$ for all n

2. $\lim_{n \to \infty} b_n = 0$

    then the series converges.

    ex) $\sum \frac{(-1)^{n-1}}{n}$

    $1/n \geq \frac{1}{n+1}$

    $\lim_{n \to \infty} 1/n = 0$

    Therefore, it converges.

14.2 Error

$|R_n| \leq b_{n+1}$

15 Comparison Test

15.1 Basics

If $b_n \geq a_n$ and $\int b_n$ converges, then $\int a_b$ converges.

   If $b_n \leq a_n$ and $\int b_n$ diverges, then $\int a_b$ diverges.

15.2 Limit Comparison

Sometimes, comparing whether a sum is larger or smaller than another is too complex. In these cases we can use:

   If $\lim_{x \to \infty} \frac{a_n}{b_n} = c$, where c is a finite number, then a_n and b_n are both divergent or convergent.

16 Alternating Series

Very simple: if an alternating series (switches between positive and negative):

1. $b_n < b_{n+1}$ (ie, do consecutive elements (ignoring alternating signs) decrease for all n?)

2. $\lim_{n \to \infty} b_n = 0$

17   Absolute vs. Conditional Convergence

A series $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

A series $\sum a_n$ is conditionally convergent if $\sum |a_n|$ diverges but $\sum a_n$ converges. (ex. alternating harmonic series)

17.1   Example

$\sum_{n=0}^{\infty} \frac{(-1)^n}{3n+5}$

Does it converge? **yes** (alternating series test)

Does $\sum |a_n|$ converge? **no** (limit comparison test)

Therefore, the sum is conditionally convergent.

18   Ratio Test

Given a series $\sum a_n$:

1. if $\lim \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the series converges absolutely

2. if $\lim \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, then the series diverges

3. if $\lim \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then the Ratio test is inconclusive

19   Root Test

Given a series $\sum a_n$:

1. if $\lim nthrootof|a_n| = L < 1$, then the series converges absolutely

2. if $\lim nthrootof|a_n| = L > 1$, then the series diverges

3. if $\lim nthrootof|a_n| = L = 1$, then the test is inconclusive

20   Power Series

A series of the form:

$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$

where the c_n are coefficients and x is a variable.

These are like polynomials, but with an infinite number of terms.

20.1 Examples

What does it mean to converge?

For a fixed value of x, does the series converge?

What are all the values of x for which the series converges?

1. $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + ...$

   This example is a geometric series, so it converges if $-1 < x < 1$.

2. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + x^2/2 + x^3/6 + x^4/24 + ...$

   We can use the ratio test to find convergence.

   $\lim_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} * \frac{n!}{x^n} \lim_{x\to\infty} \frac{x}{n+1} = 0$ for any fixed x.

   Therefore, this converges for all values of x

20.2 Power Series at a

A power series of the form

$\sum c_n(x-a)^n$

is called a *power series about a*.

All the series above have been series about 0.

Note: if $x = a$, then the series will converge. when x is "near" a, $x - a$ will be "small", so more likely to converge.

20.2.1 Radius of Convergence

Give a power series about a, there are only three possibilities:

1. The series converges at $x = 0$

2. The series converges for all values of x

3. There is a number R such that the series converges if x is within R of a, and the series diverges if x is R away from a. R is also known as the "radius of convergence".

20.2.2 Examples

Find the interval and radius of convergence of

$\sum_{n=1}^{\infty} 4^n(x-1)^n$

We can use the root test here:

$\lim_{n\to\infty} nthrootof 4^n(x-1)^n = \lim_{n\to\infty} 4(x-1)$

Therefore, converges if $|4(x-1)| < 1$, ie, if x is in the interval $(3/4, 5/4)$.

It diverges if $|4(x-1)| > 1$, ie, if x is in $(-\infty, 3/4)U(5/4, \infty)$

What if $x = 3/4, 5/4$?

$x = 3/4 => \sum 4^n(-1/4)^n = \sum(-1)^n =>$ diverges

$x = 5/4 => \sum 4^n(1/4^n) = \sum 1^n =>$ diverges.

The radius of convergence is therefore $1/4$, and the interval of convergence is $(3/4, 5/4)$.

21 Functions as Power Series

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

21.1 Examples

1. ex) $\frac{1}{1+x}$

   First way to generate new power series is to use our old known one.

   $= \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$

   Converges $|x| < 1$

2. ex) $\frac{1+x}{1-x}$

   $= (1+x)\sum_{n=0}^{\infty} x^n$

   $= \sum_{n=0}^{\infty}(1+x)x^n$

   $= \sum_{n=0}^{\infty} x^n + x^{n+1}$

   ...

   $= 1 + \sum_{n=1}^{\infty} 2x^n$

3. ex) $\frac{3}{1+x^2}$

   $= 3\frac{1}{1-(-x^2)}$

   $= 3\sum_{n=0}^{\infty}(-x^2)^n$

   $= \sum 3(-1)^n x^2 n$

21.2 Theorem

If a function f is defined by the power series $\sum c_n(x-a)^n$ with radius of convergence $R > 0$, then f is differentiable on the interval $(a-R, a+R)$ and

1. $f'(x) = \sum n c_n (x-a)^{n-1}$

2. $\int f(x)\, dx = C + \sum \frac{c_n(x-a)^{n+1}}{n+1}$

   and the radii of convergence of $f'(x)$ and $\int f(x)\, dx$ are both R.

21.2.1 Theorem Examples

1. ex) $tan^{-1}(x) =$

   Note: $\frac{d}{dx}tan^{-1}(x) = \frac{1}{1+x^2} = \sum(-1)^n x^{2n}$

   Therefore: $tan^{-1}(x) = \int \sum(-1)^n x^{2n}\, dx$

   $= \sum \int (-1)^n x^{2n}\, dx$

   $= \sum \frac{(-1)^n x^{2n+1}}{2n+1}$

   $= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ...$

21.2.2   Notes

Above theorem can be useful for computing derivatives, integrals.

1. $\int_0^1 \frac{1}{1+x^4}\,dx$

   This integral is difficult to solve with substitution, etc.

   $= \int_0^1 \sum_{n=1}^{\infty} (-1)^n x^{4n}\,dx$

   Now, just a polynomial.

   $= \sum \frac{(-1)^n x^{4n+1}}{4n+1}\big|_0^1$

   Radius of convergence is 1, so we are good.

   $= \sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1}$

   Using this, we can approximate the integral to any degree of accuracy we want with partial sums.

22   Taylor Series

22.1   Approximating Functions

We can approximate a function at a certain range by using its derivative (tangent line) at that point.

   Tangent line approx: $p_1(x) = f(a) + f'(a)(x-a) \approx f(x)$ when x is "near" a.

   In this case, $p_1(a) = f(a)$ and $p_1'(a) = f'(a)$

   By extension, a quadratic approximate of f would have:

   $p_2(a) = f(a)$
   $p_2'(a) = f'(a)$
   $p_2''(a) = f''(a)$

   We can try:

   $p_2(x) = f(a) + f'(a)(x-a) + c(x-a)^2$

   This is the same as the tangent line approximation but with a quadratic part added $(c(x-a)^2)$.

   $p_2'(x) = f'(a) + 2c(x-a)$
   $p_2''(x) = 2c$

   Since we want $p_2''(a) = f''(a)$, we can let $c = \frac{f''(a)}{2}$, and rewrite:

   $p_2(x) = f(a) + f'(a)(x-a) + \frac{f''(x)}{2}(x-a)^2$

   Following this pattern, we can continue with this pattern for more powers...

   $p_3 = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(x)}{3!}(x-a)^3$

   This patterns becomes...

22.2   Taylor Polynomials

The Taylor polynomial $p_k(x)$ of f is defined by

$p_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$

This approximates f near a. Derivatives $p_k'(a)$ through $p_k^{(k)}(a)$ will match those of f.

We can take this idea to infinity and get...

22.3   Taylor Series

If a function f can be represented as a power series at a, then its power (Taylor) series is

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$.

With infinite terms, this Taylor "approximation" should be equal to our function, if it can have a power series representation.

22.3.1   Examples

1. ex) Determine the Taylor series of e^x at $x = 0$.

   If it has a power series expansion, this is rather easy as all derivatives of any order of e^x are e^x, and $f^{(n)}(0)$ always equals 1.

   Taylor series $= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$

22.4   When do we have a Taylor series?

For any k, $f(x) = p_k(x) + R_k(x)$, where $R_k(x)$ is a remainder. This remainder is smaller if k is bigger and/or x is close to a.

Theorem: If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then $|R_k(x)| \leq \frac{M}{(k+1)!}|x-a|^{k+1}$ for values of $|x-a| \leq d$.

By extension, if $\lim_{k \to \infty} R_k(x) = 0$, then the Taylor series works.

22.5   Examples

1. ex) Determine the Taylor series of $cos(x)$ at $x = \pi$

   $f(\pi) = -1$

   $f'(\pi) = -sin(x)|_{x=\pi} - 0$

   $f''(\pi) = -cos(x)|_{x=\pi} = 1$

   $f^{(3)}(\pi) = 0$

   $f^{(4)}(\pi) = -1$

   ...

   $= -1 + 0*(x-\pi) + \frac{1}{2}(x-\pi)^2 + \frac{0}{3!}(x-\pi)^3 + \frac{-1}{4}(x-\pi)^4 + \frac{0}{5!}(x-\pi)^5 + ...$

   $= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\pi)^{2n}}{(2n)!}$

Does this work?

Yes, the limit of $R(x)$ goes to 0.

2. ex) Determine the Maclaurin series of $(1+x)^k$

Maclaurin: Taylor series at $x = 0$

If $k = 1, 2, 3...$, it is quite easy as you can just expand.

$f'(x) = k(1+x)^{k-1}$

$f''(x) = k(k-1)(1+x)^{k-2}$

...

$f^{(n)}(x) = k(k-1)...(k-n+1)(1+x)^{k-n}$

About 0:

$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)...(k-n+1)}{n!} x^n$ (Binomial expansion)

22.6 Taylor Approximations of Functions at a Point

ex) Suppose you approximate $x^{\frac{2}{3}}$ about $a = 1$ using a cubing Taylor polynomial. Find a bound on the maximum error when $0.8 \leq x \leq 1.2$.

$f(1) = 1$

$f'(1) = \frac{2}{3}x^{-\frac{1}{3}}|_{x=1} = \frac{2}{3}$

$f''(1) = -\frac{2}{9}x^{-\frac{4}{3}}|_{x=1} = -\frac{2}{9}$

$f^{(3)}(1) = \frac{8}{27}x^{-\frac{7}{3}}|_{x=1} = \frac{8}{27}$

$f(x) \approx 1 + \frac{2}{3}(x-1) - \frac{\frac{2}{9}}{2}(x-1)^2 + \frac{\frac{8}{27}}{6}(x-1)^3$

$f(\frac{2}{3}) \approx ... \approx 1.129283945$

To bound $R_3(x)$, we need $f^{(4)}(x) = \frac{56}{81}x^{-\frac{10}{3}}$

$|f^{(4)}(x)| \leq |f^{(4)}(0.8)| = 1.454576$

So, $|R_3(x)| \leq \frac{1.454576}{4!}|1.2-1|^4 = \mathbf{0.00009697}$

23 Differential Equations

What is it? An equation that involves a function and its derivative, and come up often in real-world situations.

Ex. A proposed model of population growth:

$\frac{dP}{dt} = kP$

The rate of graph of a population P is proportional to the size of the population.

Ex 2. More nuanced model of population growth:

$\frac{dP}{dt} = kP(1 - \frac{P}{M})$

23.1 Terminology

1. A differential equation is an equation that involves an unknown function and one or more of its derivatives.

   Ex) $y'' - y = 2e^x$

2. A solution to a differential equation is a function that makes that equation true.

   Ex) $y = xe^x$

3. The order of a differential equation is the highest derivative that appears.

23.2 Initial Value Problems

Ex. Verify that $f(x) = ce^{\frac{x^2}{2}}$ are solutions to the differential equation $y' = xy$, and find the solution satisfying the initial condition $y(0) = 2$.

$y = ce^{\frac{x^2}{2}}$
$y' = xce^{\frac{x^2}{2}}$
$y' = xy$
$2 = ce^0$
$c = 2$

23.3 Visualizing Solutions

Some diff equations are hard to solve....Instead of solving, we can visualize the solutions.

23.3.1 Direction Fields

Ex. $y' = y^2 - 4$

Using this, we know the slope at different points. $(0,0) : -4, (0,1) : -3, (0,2) : 0...$

We can use this to create a direction field and visualize the path the graph will take overall.

23.4 Euler's Method

A method to numerically approximate solutions.

Ex. Approximate the solution to $y' = 2 - y$ with $y(0) = 0$.
$e(0,0), y' = 2$
$e(1/2, 1), y' = 1$
$e(1, 3/2), y' = 1/2$
$e(3/2, 7/4), y' = 1/4$

23.4.1 Definition of the Method

Given the initial value problem $y' = F(x, y)$, $y(x_0) = y_0$, Euler's method with step size h approximates the solution via:

$x_n = x_{n-1} + h$

and

$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$

The smaller h, the more accurate.

Ex. Approximate the solution to $y' = y - 2x$ with $y(1) = 0$

Let's pick $h = 0.1$:

$e(1, 0) : y' = -2$

$e(1.1, 0 - 2(0.1) = e(1.1, -0.2) : y' = -2.4$

$e(1.2, -0.2 - 2.4(0.1)) = e(1.2, -0.44) : y' = ...$

...

23.5 Separable Equations

Def: A differential equation that can be written in the form:

$\frac{dy}{dx} = g(x)f(y)$

Ex: $y' = 3x\sin(y)$

23.5.1 Solving

Rewrite:

$dy = g(x)f(y)dx$

$\frac{1}{f(y)}dy = g(x)dx$

You can now integrate both sides and solve.

Ex. Solve $\frac{dy}{dx} = xy$

$\int 1/y\,dy = \int x\,dx$

$ln|y| = \frac{x^2}{2} + C$

$y = e^{\frac{x^2}{2} + C}$

...

$y = ce^{\frac{x^2}{2}}$

Ex. Solve $\frac{dy}{dt} = \frac{ln(t)}{y+1}$

$\int y + 1\,dy = \int ln(t)dt$

$\frac{y^2}{2} + y = t\ln t - t + C$

$y^2 + 2y = 2t\ln t - 2t + C$

(add one to both sides...)

$(y + 1)^2 = 2t\ln t - 2t + 1 + C$

$y = +/- \sqrt{2t\ln t - 2t + 1 + C} - 1$

24 Population Growth Models

24.1 Natural Growth

$\frac{dP}{dt} = kP$

In this model, the rate of growth is proportional to the population size.

Solve:

$\int 1/p \, dP = \int k \, dt$

$ln|P| = kt + C$

$P = e^{kt+C}$

$P = P_0 e^{kt}$ where $P(0) = P_0$, the "initial" population

24.2 Logistic Model

$\frac{dP}{dt} = kP$ where P is small, and $\frac{dP}{dt}$ is small when P approaches its carrying capacity M.

$\frac{dP}{dt} = kP(1 - \frac{P}{M})$

Solve:

$\int \frac{1}{P(1-\frac{P}{M})} = \int k \, dt$

...

$ln|P| - ln|M - P| = kt + C$

$ln|\frac{M-P}{P}| = -kt + C$

$\frac{M-P}{P} = e^{-kt+C}$

$\frac{M}{P} - 1 = Ae^{-kt}$

$P = \frac{M}{1+Ae^{-kt}}$

24.3 Other Models

$\frac{dP}{dt} = kP(1 - \frac{P}{M}) - c$ (constant rate of loss)

$\frac{dP}{dt} = kP(1 - \frac{P}{M})(1 - \frac{m}{P})$ (if $P < m$, pop. decreases)

25 Linear Equations

A *first-order linear* differential equation is one that can be written in the form:

$\frac{dy}{dx} + P(x)y = Q(x)$

Linear in y and y', not necessarily in x.

ex) $y' + sin(x)y = e^x$

25.1 Example

$y' + \frac{1}{x}y = x^2$

$xy' + y = x^3$

$(xy)' = x^3$

$xy = \frac{x^4}{4} + C$

$y = \frac{x^3}{4} + C/x$

25.2 General Method of Solving

$\frac{dy}{dx} + P(x)y = Q(x)$

We want to multiply by something, $I(x)$, so that $(I(x)\frac{dy}{dx} + I(x)P(x)y) = (I(x)y)'$

Then, we can solve:

$(I(x)y)' = I(x)Q(x)$

$I(x)y = \int I(x)Q(x)dx$

$y = \frac{1}{I(x)} \int I(x)Q(x)dx$

What is $I(x)$, or the "integrating factor", then?

IF $I(x)y' + I(x)P(x)y = (I(x)y)'$

then $I(x)y' + I(x)P(x)y = I'(x)y + I(x)y'$

$I(x)P(x) = \frac{dI}{dx}$

$P(x)dx = \frac{1}{I}dI$

$\int P(x)dx = ln|I|$

$\mathbf{e}^{\int \mathbf{P(x)dx}} = \mathbf{I}$

25.2.1 Example

ex) $y' + y = sin(e^x)$

$P(x) = 1$

$I(x) = e^{\int P(x)dx} = e^x$

$e^x y' + e^x y = e^x sin(e^x)$

$(e^x y)' = e^x sin(e^x)$

$\int (e^x y)' dx = \int e^x sin(e^x)dx$

$e^x y = -cos(e^x) + C$

$y = \frac{-cos(e^x)}{e^x} + \frac{C}{e^x}$

26 Predator-Prey System

26.1 Assumptions of the System

1. If no predators, the population of prey will grow unconstrained.

2. If no prey, the population of predators will decline.

3. Death rate of prey depends on number of interactions between prey and predator.

4. Growth rate of predators depends on number of interactions between prey and predator.

5. Number of interactions of prey and predator is proportional to the product of their populations.

26.2   The Model

$\frac{dR}{dt} = kR - aRW$ and $\frac{dW}{dt} = -rW + bRW$

In the first equation: The growth rate of rabbits (R) is proportional to the number of rabbits, minus the number of rabbits eaten (aRW).

In the second equation: The growth rate of wolves (W) is proportional to the number of wolves and rabbits, and will decline if there are no rabbits, plus the number of wolves eating rabbits (bRw).

26.3   Example

$\frac{dR}{dt} = 0.03R - 0.0004RW$ and $\frac{dW}{dt} = -0.01W + 0.002RW$

* *Note:* if $R = 0$ and $W = 0$, then $\frac{dR}{dt} = 0$ and $\frac{dW}{dt} = 0$. We can then call $(0,0)$ and **equilibrium point**.

By the chain rule: $\frac{dW}{dt} = \frac{dW}{dR} * \frac{dR}{dt}$

Therefore, $\frac{dW}{dR} = \frac{-0.01W + 0.002RW}{0.03R - 0.0004RW}$

If graphed on a phase plane, you can see a near-cyclical relationship between the two populations, with an equilibrium point near the center.

26.4   Finding the Equilibrium Point

This is the point where both derivatives are 0.

$0 = 0.03R - 0.0004RW$

$0 = R(0.03 - 0.0004W)$

W = 75

$0 = -0.01W + 0.002RW$

R = 5

26.5   Visualizing Individual Populations

Other than just graphing the relationship between the two populations, we can also graph the individual populations of the rabbits and wolves fairly easily.

27   Second-Order Linear Equations

A second-order homogeneous linear differential equation with constant coefficients is one that can be written in the form:

$ay'' + by' + cy = 0$

where a, b, c are real constants.

second-order: involves y'

homogeneous: $= 0$

linear: y'', y', y

constant coefficients: a, b, c are real numbers

27.1 Solutions

1. Fact 1: if $y_1(x)$ and $y_2(x)$ are two solutions to

$ay'' + by' + cy = 0$,

then so are $c_1 y_1(x) + c_2 y_2(x)$ are any real numbers c_1, c_2

2. Fact 2: if $y_1(x)$ and $y_2(x)$ are two linearly independent (not multiples of each other) solutions to

$ay'' + by' + cy = 0$,

then all solutions are of the form $c_1 y_1(x) + c_2 y_2(x)$ for some real numbers c_1, c_2.

27.2 Guessing a Solution

ex) $y'' + 3y' - 4y = 0$

Maybe e^{rx}?

$y = e^{rx}, y' = re^{rx}, y'' = r^2 e^{rx}$

$r^2 e^{rx} + 3re^{rx} - 4e^{rx} = 0$

$e^{rx}(r^2 + 3r - 4) = 0$

$e^{rx}(r + 4)(r - 1) = 0$

$r = 1, -4$

By our theorem, our general solution is $c_1 e^x + c_2 e^{-4x}$

27.3 General Method

$ay'' + by' + cy = 0$

Guess e^{rx}, etc

$e^{rx}(ar^2 + br + c) = 0$

So e^{rx} is a solution if $ar^2 + br + c = 0$ (auxiliary equation).

27.3.1 Cases

The auxiliary equation $ar^2 + br + c = 0$ may have:

1. two distinct real roots, r_1 and r_2. The general solution is $c_1 e^{r_1 x} + c_2 e^{r_2 x}$, happens when discriminant is greater than 0.

2. one single real root, r. The general solution is $c_1 e^{rx} + c_2 x e^{rx}$, happens when discriminant is equal to 0.

3. two complex roots, r_1 and r_2. The general solution is $e^{\alpha x}(c_1 cos(\beta x) + c_2 sin(\beta x)$, happens when discriminant is less than 0.

Example of single real root: $y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$(r - 2)^2 = 0$

$r = 2$

We know that e^{2x} is a solution. We can also see (through some math) that xe^{rx} is also a solution.

Example of two complex roots: $y'' + 4y' + 5y = 0$

$r^2 + 4r + 5 = 0$

$r = \frac{-4 \pm \sqrt{16-20}}{2}$

$r = -2 \pm i$

We could say that the solution is $c_1 e^{(-2+i)x} + c_2 e^{(-2-i)x}$

Let's simplify this with Euler's Formula, $e^{i\theta} = cos(\theta) + isin(\theta)$

General solution: $c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$

$= c_1 e^{\alpha x}(cos(\beta x) + isin(\beta x)) + c_2 e^{\alpha x}(cos(-\beta x) + isin(-i\beta x))$

$= c_1 e^{\alpha x}cos(\beta x) + ic_1 e^{\alpha x}sin(\beta x) + c_2 e^{\alpha x}cos(\beta x) - ic_2 e^{\alpha x}sin(\beta x)$

$= e^{\alpha x}(c_1 + c_2)cos(\beta x) + e^{\alpha x}i(c_1 - c_2)sin(\beta x)$

say $C_1 = c_1 + c_2$ and $C_2 = c_1 - c_2$

$= C_1(e^{\alpha x}cos(\beta x) + C_2 e^{\alpha x}sin(\beta x)$

$= e^{\alpha x}(C_1 cos(\beta x)) + C_2 sin(\beta x)$

27.4   Boundary Value Problems

ex) Find the solution to $y'' - 6y' + 9y = 0$ such that $y(0) = 2$ and $y(3) = 1$

$(r - 3)^2 = 0, r = 3$

General solution: $c_1 e^{3x} + c_2 x e^{3x}$

$y(0) = 2, c_1 = 2$

$y(3) = 1, 2e^9 + 3c_2 e^9 = 1, c_2 = \frac{e^{-9}-2}{3}$

27.5   NON-Homogeneous Second-Order Linear Differential Equations with Constant Coefficients

$ay'' + by'' + cy = G(x)$

Complementary equation: $ay'' + by' + cy = 0$, solution: $y_c(x)$

Say we can find a particular solution, $y_p(x)$

Theorem: The general solution to the non-homogeneous equation is given by $y_c(x) + y_p(x)$

27.5.1   Solving; Two methods

1. Undetermined Coefficients

   (a) easier, straightforward

   (b) like guess and check ... kind of?

   (c) doesn't always work

2. harder

3. works more widely

27.5.2 Undetermined Coefficients

1. Take a reasonable guess at $y_p(x)$, but leave the coefficients undetermined (good guess? Same form as $G(x)$)

2. Solve to find what the coefficients should be

ex) $y'' - y = x^2 + 2$
Guess: $y_p(x) = Ax^2 + Bx + C$
$y_p'(x) = 2Ax + B$
$y_p''(x) = 2A$
$2A - (Ax^2 + Bx + C) = x^2 + 2$
$-Ax^2 - Bx + 2A - C = x^2 + 2$
$A = -1, B = 0, C = -4$
Therefore, we claim that $-x^2 - 4$ is a particular solution to this equation.
Complementary equation: $r^2 - 1 = 0$, $r = \pm 1$, $c_1 e^x + c_2 e^{-x}$
Therefore, our general solution is $c_1 e^x + c_2 e^{-x} + (-x^2 + 4)$
ex) $y'' + y = sin(x)$
Watch out: homogeneous equation has solution of the form $c_1 cos(x) + c_2 sin(x)$
Normally, we would try $y_p(x) = Asin(x) + Bcos(x)$, but since is the same form as the homogeneous solution, it won't work. Instead, try multiplying by x; $y_p = Axsin(x) + Bxcos(x)$

27.5.3 Variation of Parameters

Solution to complementary: $c_1 y_1(x) + c_2 y_2(x)$
We can view c_1 and c_2 as our parameters, and we can vary them to look for a particular solution of the form $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$
Then $y_p'(x) = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$
We need to satisfy 2 conditions: 1) u_1, u_2 make $u_1y_1 + u_2y_2$ a solution and 2) say $u_1'y_1 + u_2'y_2 = 0$, to make life easier
$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$
(Assume y_p is a solution)
$a(u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + b(u_1y_1' + u_2y_2') + c(u_1y_1 + u_2y_2) = G$
$u_1(ay_1'' + by_1' + cy_1) + u_2(ay_2'' + by_2' + cy_2) + a(u_1'y_1' + u_2'y_2') = G$
However, since $ay_1'' + by_1' + cy_1$ and $ay_2'' + by_2' + cy_2$ are solutions to the homogeneous equation, they just equal 0.
$a(u_1'y_1' + u_2'y_2') = G$
We also assume that $u_1'y_1 + u_2'y_2 = 0$, and if we can satisfy both equations, we can find a particular solution.
Example: $y'' - 2y' - 3y = e^x$
Homogeneous: $(r - 3)(r + 1) = 0$, $r = 3, -1$
$y_c = c_1 e^{3x} + c_2 e^{-x}$
Suppose $y_p = u_1 e^{3x} + u_2 e^{-x}$
$y_p' = u_1' e^{3x} + 3u_1 e^{3x} + u_2' e^{-x} - u_2 e^{-x}$
Let $u_1' e^{3x} + u_2' e^{-x} = 0$

$y'' = 3u_1'e^{3x} + 9u_1e^{3x} - u_2'e^{-x} + u_2e^{-x}$
$3u_1'e^{3x} + 9u_1e^{3x} - u_2'e^{-x} + u_2e^{-x} - 2(3u_1e^{3x} - u_2e^{-x}) - 3(3u_1e^{3x} - u_2e^{-x}) = e^x$
$u_1(9e^{3x} - 6e^{3x} - 3e^{3x}) + u_2(e^{-x} + 2e^{-x} - 3e^{-x}) + 3u_1'e^{3x} - u_2'e^{-x} = e^x$
$3u_1'e^{3x} - u_2'e^{-x} = e^x$
solve with both equations...
$y_p(x) = \frac{e^{-3x}}{8}e^{3x} + \frac{-e^{2x}}{8}e^{-x} = \frac{-e^x}{4}$
General solution: $c_1e^{3x} + c_2e^{-x} - \frac{e^x}{4}$

28 Second-Order Linear Equations Applications: Springs

28.1 Hooke's Law

If a spring is stretched or compressed x units from its nature length, the corresponding force exerted is proportional to the distance x, Force $= -kx$, where k is called the *spring constant*.

28.2 Newton's Second Law

$f = ma$

Since acceleration is the second derivative of distance, we can combine this with Hooke's Law: $m\frac{d^2x}{dt^2} = -kx$, or $mx'' + kx = 0$, a homogeneous differential equation.

28.3 Simple Harmonic Motion

$mx'' + kx = 0$
$\quad mr^2 + k = 0$
$\quad r^2 = -k/m$
$\quad r = \pm i\sqrt{\frac{k}{m}}$

Solution: $c_1cos(\sqrt{\frac{k}{m}}t) + c_2sin(\sqrt{\frac{k}{m}}t)$

Often written as $x(t) = Acos(\omega t + \delta)$, where $\omega = \sqrt{\frac{k}{m}}$, $A = \sqrt{c_1^2 + c_2^2}$, and $cos(\delta) = \frac{c_1}{A}$, $sin(\delta) = \frac{-c_2}{A}$. This form makes it clearer that the function has a frequency and amplitude, and is periodic.

28.4 Example

ex) A spring with a 5-kg mass is kept stretched 0.5 m beyond its natural length by a force of 40 N. The spring starts at its equilibrium point and is given an initial velocity of 0.2 m/s. Find the position of the mass at any time t.
$\quad 40 = k(0.5)$
$\quad 80 = k$
$\quad 5x'' = -80x$

$5x'' + 80x = 0$

$5r^2 + 80 = 0$

...

$r = \pm 2i$

$\mathbf{x(t) = c_1 cos(2t) + c_2 sin(2t)}$

$x(0) = 0$

$0 = c_1 cos(0) + c_2 sin(0)$

$\mathbf{0 = c_1}$

$x'(t) = -2c_1 sin(2t) + 2c_2 cos(2t)$

$x'(0) = 0.2 => 2c_2 = 0.2$

$\mathbf{c_2 = 0.1}$

Solution: $x(t) = 0.2sin(2t)$

28.5 Damping Force

Damping force is proportional to the velocity of the mass:

Damping Force $= -c\frac{dx}{dt}$

Adding this to our earlier equation:

$F = ma = -kx - c\frac{dx}{dt}$

$mx'' + cx' + kx = 0$

$mr^2 + cr + k = 0$

$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$

We have to look at the three possible cases for the discriminant here:

1. over-damping: $c^2 - 4mk > 0$

   roots will be less than 0, giving us exponential decay $(x = c_1 e^{r_1 t} + c_2 e^{r_2 t})$

2. critical damping: $c^2 - 4mk = 0$

   gets back to equilibrium as quickly as possible $(x = c_1 e^{rt} + c_2 x e^{rt})$

3. under-damping $c^2 - 4mk < 0$

   oscillates $(x = e^{-rt}(c_1 cos(\omega t) + c_2 sin(\omega t))$

29   Using Power Series

29.1   Why?

It is a lot easier to work with and find solutions that can't be expressed as function when looking at a function as a power series when solving differential equations.

29.2 Examples

1. $y'' - y = 0$

   Assume that $y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 ...$

   $y' = \sum_{n=1}^{\infty} n c_n x^{n-1} c_1 2 c_2 x + 3 c_3 x^2 ...$

   $\sum_{n=1}^{\infty} n c_n^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$

   $\sum_{n=0}^{\infty} (n+1) c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n = 0$

   $\sum ((n+1) c_{n+1} - c_n) x^n = 0$

   Therefore, $(n+1) c_{n+1} - c_n = 0$ for all n.

   Value for c_0 implies values for the rest; a recursive relationship.

   Say: $c_0 = 1$

   $c_1 = 1$

   $c_2 = \frac{1}{2}$

   $c_3 = \frac{1}{3*2}$

   $c_4 = \frac{1}{4*3*2}$

   $\mathbf{c_n = \frac{1}{n!}}$

   $\mathbf{y = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x}$

2. $y'' - xy' - y = 0$

   For this problem, our normal methods don't work.

   Say: $y = \sum_{n=0}^{\infty} c_n x^n$

   $y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$

   $y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$

   $\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - x \sum_{n=1}^{\infty} n c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$

   (simplify and rewrite starting at $n = 0$)

   $\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$

   $\sum_{n=0}^{\infty} ((n+1)(n+1) c_{n+2} - n c_n - c_n) x^n = 0$

   $(n+2)(n+1) c_{n+2} - (n+1) c_n = 0$

   $c_{n+2} = \frac{c_n}{n+2}$ *(This is our recursive relationship)*

   $c_0 = c_0$

   $c_1 = c_1$

   $c_2 = \frac{c_0}{2}$

   $c_3 = \frac{c_1}{3}$

   $c_4 = \frac{c_0}{8}$

   $c_5 = \frac{c_1}{15}$

   You can separate this out by even vs odd terms.

$$y = \sum_{n=0}^{\infty} \frac{c_0 x^{2n})}{(2n)(2n-2)(2n-4)...(2)} + \sum_{n=0}^{\infty} \frac{c_1 x^{2n+1}}{(2n+1)(2n-1)...(1)}$$

$$= \sum_{n=0}^{\infty} \frac{c_0 x^{2n}}{2^n n!} - \sum_{n=0}^{\infty} \frac{c_1 x^{2n+1}}{\frac{(2n+1)!}{2^n n!}}$$

$$= \sum_{n=0}^{\infty} \frac{c_0 (\frac{x^2}{2})^n}{n!} + ...$$

$$= c_0 e^{\frac{x^2}{2}} + \sum_{n=0}^{\infty} \frac{c_1 x^{2n+1}}{(2n+1)!}$$

Although we were able to find an elementary function for our left side, there is none for our right side, and therefore must be left in power series notation. In practice, we could approximate this value to any degree of accuracy with previously discussed methods.