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Analysis I

MATH254

<u>Course Outline:</u>

Fundamentals of set theory. Properties of the reals. Limits, limsup, liminf. Continuity. Functions. Differentiation.

<u>*References:*</u>

Understanding Analysis, *Abbott;* Introduction to Real Analysis, *Bartle;* Analysis I, *Tao*

Based on Lectures from Fall, 2023 by Prof. Vojkan Jaksic.

Contents

1 Logic, Sets, and Functions

1.1 Mathematical Induction & The Naturals

The **natural numbers**, $\mathbb{N} = \{1, 2, 3, \dots\}$, are specified by the 5 **Peano Axioms**:

(1) $1 \in \mathbb{N}$ [1]

(2) every natural number has a successor in $\mathbb{N}$

(3) 1 is not the successor of any natural number

(4) if the successor of x is equal to the successor of y, then x is equal to y [2]

(5) **the axiom of induction**

[1]using 0 instead of 1 is also valid, but we will use 1 here, and throughout the rest of course.

[2]axioms (2)-(4) can be equivalently stated in terms of a successor function $s(n)$ more rigorously, but won't here

The **Axiom of Induction** (AI), can be stated in a number of ways.

↪ **Axiom 1.1: AI.i**

Let $S \subseteq \mathbb{N}$ with the properties:

(a) $1 \in S$

(b) if $n \in S$, then $n + 1 \in S$[3]

then $S = \mathbb{N}$.

[3]*(a)* is called the **inductive base**; *(b)* the **inductive step**. All AI restatements are equivalent in having both of these, and only differentiate on their specific values.

⊛ **Example 1.1**

Prove that, for every $n \in \mathbb{N}$, $1 + 2 + \cdots + n = \frac{n(n+1)}{2} (\equiv (1))$

Proof (via AI.i). Let S be the subset of $\mathbb{N}$ for which (1) holds; thus, our goal is to show $S = \mathbb{N}$, and we must prove *(a)* and (b) of AI.i.

- by inspection, $1 \in S$ since $1 = \frac{1(1+1)}{2} = 1$, proving *(a)*
- assume $n \in S$; then, $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ by definition of S. Adding $n + 1$

to both sides yields:

$$
\begin{aligned}
1+2+\cdots+n+(n+1) &= \frac{n(n+1)}{2}+(n+1) && (1)\\
&= (n+1)(\frac{n}{2}+1) && (2)\\
&= \frac{(n+1)(n+2)}{2} && (3)\\
&= \frac{(n+1)((n+1)+1)}{2} && (4)
\end{aligned}
$$

Line (4) is equivalent to statement (1) (substituting n for $n+1$), and thus if $n \in S$, then $n+1 \in S$ and *(b)* holds. Thus, by AI.i, $S = \mathbb{N}$ and $1+2+\cdots+n = \frac{n(n+1)}{2}$ holds $\forall\, n \in \mathbb{N}$. ■

⊛ **Example 1.2**

Prove (by induction), that for every $n \in \mathbb{N}$, $1^3+2^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$.

Proof. Follows a similar structure to the previous example. Let S be the subset of $\mathbb{N}$ for which the statement holds. $1 \in S$ by inspection (*(a)* holds), and we prove *(b)* by assuming $n \in S$ and showing $n+1 \in S$ (algebraically). Thus, by AI.i, $S = \mathbb{N}$ and the statement holds $\forall\, n \in \mathbb{N}$. ■

This can also be proven directly (Gauss' method).

Proof (Gauss' method). Let $A(n) = 1+2+3+\cdots+n$. We can write $2 \cdot A(n) = 1+2+3+\cdots+n+1+2+3+\cdots+n$. Rearranging terms (1 with n, 2 with $n-1$, etc.), we can say $2 \cdot A(n) = (n+1)+(n+1)+\cdots$, where $(n+1)$ is repeated n times; thus, $2 \cdot A(n) = n(n+1)$, and $A(n) = \frac{n(n+1)}{2}$. ■

↪ **Axiom 1.2: AI.ii**

Let $S \subseteq \mathbb{N}$ s.t.

(a) $m \in S$

(b) $n \in S \implies n+1 \in S$

then $\{m, m+1, m+2, \dots\} \subseteq S$.

⊛ **Example 1.3**

Using AI.ii, prove that for $n \geq 2$, $n^2 > n+1$.

Proof. Let $S \subseteq \mathbb{N}$ be the set of n for which the statement holds. $n = 2 \implies 4 > 3$, so the base case holds. Consider $n^2 > n+1$ for some $n \geq 2$. Then, $(n+1)^2 = n^2 + 2n + 1 > n + 1 + 2n + 1 = 3n + 2 > 2n + 2 > n + 2$, hence $S = \{2, 3, 4, \cdots\}$ (all $n \geq 2$). ■

↪ **Axiom 1.3: Principle of Complete Induction, AI.iii**

Let $S \subseteq \mathbb{N}$ s.t.

(a) $1 \in S$

(b) if $1, 2, \ldots, n-1 \in S$, then $n \in S$

then $S = \mathbb{N}$.

Finally, combining AI.ii and AI.iii;

↪ **Axiom 1.4: AI.iv**

Let $S \subseteq \mathbb{N}$ s.t.:

(a) $m \in S$

(b) if $m, m+1, \ldots, m+n \in S$, then $m+n+1 \in S$

then $\{m, m+1, m+2, \ldots\} \subseteq S$.

↪ **Theorem 1.1: Fundamental Theorem of Arithmetic**

Every natural number n can be written as a product of one or more primes. [4]

[4] 1 is not a prime number

Proof of theorem 1.1. Let S be the set of all natural numbers that can be written as a product of one or more primes. We will use AI.iv to show $S = \{2, 3, \ldots\}$.

- *(a)* holds; 2 is prime and thus $2 \in S$
- suppose that $2, 3, \ldots, 2+n \in S$. Consider $2 + (n+1)$:

- if $2+(n+1)$ is *prime*, then $2+(n+1) \in S$, as all primes are products of 1 and themselves and are thus in S by definition.
- if $2+(n+1)$ is *not prime*, then it can be written as $2+(n+1) = a \cdot b$ where $a, b \in \mathbb{N}$, and $1 < a < 2+(n+1)$ and $1 < b < 2+(n+1)$. By the definition of S, $a, b \in S$, and can thus be written as the product of primes. Let $a = p_1 \cdot \dots \cdot p_l$ and $b = q_1 \cdot \dots \cdot q_j$, where the p's and q's are prime and $l, j \geq 1$. Then, $a \cdot b$ is a product of primes, and thus so is $2+(n+1)$. Thus, $2+(n+1) \in S$, and by AI.iv, $S = \{2, 3, 4, \dots\}$

■

1.2 Extensions: Integers, Rationals, Reals

Consider the set of naturals $\mathbb{N} = \{1, 2, 3, \dots\}$. Adding 0 to $\mathbb{N}$ defines $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We define the **integers** as the set $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, or the set of all positive and negative whole numbers.

Within $\mathbb{Z}$, we can define multiplication, addition and subtraction, with the neutrals of 1 and 0, respectively. However, we cannot define division, as we are not guaranteed a quotient in $\mathbb{Z}$. This necessitates the **rationals**, $\mathbb{Q}$. We define

$$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{Z}, q \neq 0\}.$$

On $\mathbb{Q}$, we have the familiar operations of multiplication, addition, subtraction and properties of associativity, distributivity, etc. We can also define division, as $\frac{\frac{p}{q}}{\frac{p'}{q'}} = \frac{pq'}{qp'}$.

We can also define a relation $<$ between fractions, such that

- $x < y$ and $y < z \implies x < z$
- $x < y \implies x + z < y + z$

$\mathbb{Q}$, together with its operations and relations above, is called an **ordered field**.

1.2.1 The Insufficiency of the Rationals

We can consider historical reasoning for the extension of $\mathbb{Q}$ to $\mathbb{R}$. Consider a right triangle of legs a, b and hypotenuse c. By the Pythagorean Theorem, $a^2 + b^2 = c^2$. Consider further the case there $a = b = 1$, and thus $c^2 = 2$. Does c exist in $\mathbb{Q}$?

→ **Proposition 1.1**

$c^2 = 2$, $c \notin \mathbb{Q}$.

Proof of proposition 1.1. Suppose $c \in \mathbb{Q}$. We can thus write $c = \frac{p}{q}$, where[5] $p, q \in \mathbb{N}$, and p, q share no common divisors, ie they are in "simplest form". Notably, p and q cannot *both* be even (under our initial assumption), as they would then share a divisor of 2. We write

$$c = \frac{p}{q}$$
$$c^2 = 2 = \frac{p^2}{q^2}$$
$$2q^2 = p^2$$

$p \in \mathbb{N} \implies p^2 \in \mathbb{N}$, and thus p^2, and therefore[6] p, must be divisible by 2 ($\implies p$ even). Therefore, we can write $p = 2p_1, p_1 \in \mathbb{N}$, and thus $2q^2 = (2p_1^2)^2 \implies q^2 = 2p_1^2$. By the same reasoning, q must now be even as well, contradicting our initial assumption that p and q share no common divisors. Thus, $c \notin \mathbb{Q}$. ■

[5] Note that in the definition of $\mathbb{Q}$, p, q are defined to be in $\mathbb{Z}$; however, as we are using a geometric argument, we can assume $c > 0 \implies \text{Sign}(p) = \text{Sign}(q)$, and we can just take $p, q \in \mathbb{N}$ for convenience and wlog.

[6] $\sqrt{\text{even}} = \text{even}$

1.3 Sets & Set Operations

- $A \cup B = \{x : x \in A \text{ or } x \in B\}$
- $A \cap B = \{x : x \in A \text{ and } x \in B\}$
- $\bigcup_{i=1}^{\infty} A_n = \bigcup_{n \in \mathbb{N}} A_n = \{x : x \in A_n \text{ for some } n \in \mathbb{N}\}$
- $\bigcap_{i=1}^{\infty} A_n = \bigcap_{n \in \mathbb{N}} A_n = \{x : x \in A_n \,\forall\, n \in \mathbb{N}\}$
- $A^C = \{x : x \in X \text{ and } x \notin A\}$[7]

[7] X is often omitted if it is clear from context.

→ **Theorem 1.2: De Morgan's Theorem(s)**

Let A, B be sets. Then,

$$(a) \qquad (A \cap B)^C = A^C \cup B^C$$

and

$$(b) \qquad (A \cup B)^C = A^C \cap B^C.$$

Proof of theorem 1.2. (b) *(A similar argument follows…)* ■

↪ **Proposition 1.2**

$$(a)\ \left(\bigcap_{n=1}^{\infty} A_n\right)^C = \bigcup_{n=1}^{\infty} A_n^C$$
$$(b)\ \left(\bigcup_{n=1}^{\infty} A_n\right)^C = \bigcap_{n=1}^{\infty} A_n^C$$

Proof of proposition 1.2. Consider Proposition (b). Working from the left-hand side, we have

$$\begin{aligned}\left(\bigcup_{n=1}^{\infty} A_n\right)^C &= \{x : x \notin \bigcup A_n\} \\ &= \{x : x \notin A_n \ \forall\ n \in \mathbb{N}\} \\ &= \bigcap \{x : x \notin A_n\} \\ &= \bigcap A_n^C\end{aligned}$$

(a) can be logically deduced from this result. Consider the RHS, $\bigcup A_n^C$. Taking the complement:

$$\begin{aligned}\left(\bigcup A_n^C\right)^C &\overset{\text{via (b)}}{=} \bigcap A_n^{C^C} \\ &= \bigcap A_n\end{aligned}$$

Taking the complement of both sides, we have $\bigcup A_n^C = \left(\bigcap A_n\right)^C$, proving (a). ■

1.4 Functions

↪ **Definition 1.1**

Let A, B be sets. A *function* f is a rule assigned to each $x \in A$ a corresponding unique element $f(x) \in B$. We denote

$$f : A \to B.$$

↪ **Definition 1.2**

The *domain* of a function $f : A \to B$, denoted $\mathrm{Dom}(f) = A$. The *range* of f, denoted

$\text{Ran}(f) = \{f(x) : x \in A\}$. Clearly, $\text{Ran}(f) \subseteq B$, though equality is not necessary.

⊛ **Example 1.4**

The function $f(x) = \sin x$, $f : \mathbb{R} \to [-1, 1]$. Here, $\text{Dom}(f) = \mathbb{R}$, and $\text{Ran}(f) = [-1, 1]$.

⊛ **Example 1.5: Dirichlet Function**

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \notin \mathbb{Q} \end{cases}$. Despite not having a true "explicit" formula, so to speak, this is still a valid function (under modern definitions).

1.4.1 Properties of Functions

↪ **Proposition 1.3**

Let $f : A \to B$, $C \subseteq A$, $f(C) = \{f(x) : x \in C\}$. We claim $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$.

Proof. We will prove this by showing (1) $\subseteq$ and (2) $\supseteq$.

(1) $y \in f(C_1 \cup C_2) \implies$ for some $x \in C_1 \cup C_2, y = f(x)$. This means that either for some $x \in C_1, y = f(x)$, or for some $x \in C_2, y = f(x)$. This implies that either $y \in f(C_1)$, or $y \in f(C_2)$, and thus y ***must*** be in their union, ie $y \in C_1 \cup C_2$.

(2) $y \in f(C_1) \cup f(C_2) \implies y \in f(C_1)$ or $y \in f(C_2)$. This means that for some $x \in C_1, y = f(x)$, or for some $x \in C_2, y = f(x)$. Thus, x ***must*** be in $C_1 \cup C_2$, and for some $x \in C_1 \cup C_2, y = f(x) \implies y \in f(C_1 \cup C_2)$.

(1) and (2) together imply that $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$. ■

⊛ **Example 1.6**

Let $A_n = 1, 2, \ldots$ be a sequence of sets. Prove that $f(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f(A_n)$.

Proof. Let $y \in f(\bigcup_{n=1}^{\infty} A_n)$. This implies that $\exists x \in \bigcup_{n=1}^{\infty} A_n$ s.t. $f(x) = y$. This implies that $x \in A_n$ for some n, and $y \in f(A_n)$ for that same "some" n, and thus y must be in the union of all possible $f(A_n)$, ie $y \in \bigcup f(A_n)$. This shows $\subseteq$, use similar logic for the reverse. ■

↪ **Proposition 1.4**

$f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$ [8]

[8] NB: the reverse is not always true, ie these sets are not always equal; "lack" of equality is more "common" than not.

Proof. $y \in f(C_1 \cap C_2) \implies$ for some $x \in C_1 \cap C_2, y = f(x)$. This implies that for some $x \in C_1, y = f(x)$ **and** for some $x \in C_2, y = f(x)$. Note that this does *not* imply that these x's are the same, ie this reasoning is not reversible as in the previous union case. This implies that $y \in f(C_1)$ and $y \in f(C_2) \implies y \in f(C_1) \cap f(C_2)$. ■

⊛ **Example 1.7**

Prove that if $A_n, n = 1, 2, \ldots$, $f(\bigcap_{n=1}^{\infty} A_n) \subseteq \bigcap_{n=1}^{\infty} f(A_n)$.

Proof (Sketch). Use the same idea as in example 1.6, but, naturally, with intersections. ■

⊛ **Example 1.8**

Take $f(x) = \sin x$, $A = \mathbb{R}, B = \mathbb{R}$, and take $C_1 = [0, 2\pi], C_2 = [2\pi, 4\pi]$. Then, $f(C_1) = [-1, 1]$, and $f(C_2) = [-1, 1]$. But $C_1 \cap C_2 = \{2\pi\}$; $f(\{2\pi\}) = \{\sin 2\pi\} = \{0\}$, and thus $f(C_1 \cap C_2) = \{0\}$, while $f(C_1) \cap f(C_2) = [-1, 1]$, as shown in proposition 1.4.

↪ **Definition 1.3: Inverse Image of a Set**

Let $f : A \to B$ and $D \subseteq B$. The *inverse image* of D by F is denoted $f^{-1}(D)$[9] and is defined as

$$f^{-1}(D) = \{x \in A : f(x) \in D\}.$$

[9] Note that this is **not** equivalent to the typical definition of an inverse *function*; f^{-1} may not exist

⊛ **Example 1.9**

$A = [0, 2\pi], B = \mathbb{R}, f(x) = \sin x, D = [0, 1]$.

$$f^{-1}(D) = \{x \in A : f(x) \in D\} = \{x \in [0, 2\pi] : \sin(x) \in [0, 1]\} = [0, \pi].$$

↪ **Proposition 1.5**

Given function f and sets D_1, D_2,

(a) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$

(b) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$[10]

[10] Just see next proposition; if you really need convincing, just use 2 rather than ∞ as the upper limit of the unions/intersections and use the same proof.

↪ **Proposition 1.6:** ⋆

Let $A_n, n = 1, 2, 3 \ldots$. Then,

(a) $f^{-1}(\bigcup_{n=1}^{\infty} A_n) = \bigcup_{n=1}^{\infty} f^{-1}(A_n)$

(b) $f^{-1}(\bigcap_{n=1}^{\infty} A_n) = \bigcap_{n=1}^{\infty} f^{-1}(A_n)$

Proof. [11]

(a)

$$\begin{aligned} x \in f^{-1}(\bigcup_{n=1}^{\infty} A_n) &\iff f(x) \in \bigcup_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for some } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for some } n \in \mathbb{N} \\ &\iff x \in \bigcup_{n=1}^{\infty} f^{-1}(A_n) \end{aligned}$$

(b)

$$\begin{aligned} x \in f^{-1}(\bigcap_{n=1}^{\infty} A_n) &\iff f(x) \in \bigcap_{n=1}^{\infty} A_n \\ &\iff f(x) \in A_n \text{ for all } n \in \mathbb{N} \\ &\iff x \in f^{-1}(A_n) \text{ for all } n \in \mathbb{N} \\ &\iff x \in \bigcap_{n=1}^{\infty} f^{-1}(A_n)^{12} \end{aligned}$$

■

[12] This is a "proof by definitions" as I like to call it.

[12] Similar proof can be used to prove proposition 1.5, less generally.

Remark 1.1. *$f : A \to B$, $A_1 \subseteq A$. Given $f(A_1^C)$ and $f(A_1)^C$, there is* ***no general relation*** *between the two.*

For instance, take $A = [0, 6\pi], B = [-1, 2], C = [0, 2\pi]$, and $f(x) = \sin x$. Then, $f(C) = [-1, 1]$, and $f(C^C) = f([-1, 0)) = [-1, 1]$, but $f(C)^C = [-1, 1]^C = (1, 2]$, and $f(C^C) \neq f(C)^C$; in fact, these sets are disjoint.

↪ **Proposition 1.7**

Let $f : A \to B$ and let $D \subseteq B$. Then $f^{-1}(D^C) = [f^{-1}(D)]^C$.

Proof.

$$f^{-1}(D^C) = \{x : f(x) \in D^C\} = \{x : f(x) \notin D\}$$
$$[f^{-1}(D)]^C = [\{x : f(x) \in D\}]^C = \{x : x \notin f^{-1}(D)\} = \{x : f(x) \notin D\}$$

■

1.5 Reals

↪ **Axiom 1.5: Of Completeness**

Any non-empty subset of $\mathbb{R}$ that is bound from above has at least one upper bound (also called the supremum).

In other words; let $A \subseteq \mathbb{R}$ and suppose A is bounded from above (A has at a least upper bound). Then $\sup(A)$ exists.

Real numbers, algebraically, have the same properties as the rationals; we have addition, multiplication, inverse of non-zero real numbers, and we have the relation $<$. All together, $\mathbb{R}$ is an ordered field.

↪ **Definition 1.4**

Let $A \subseteq \mathbb{R}$. A number $b \in \mathbb{R}$ is called an **upper bound** for A if for any $x \in A$, $x \leq B$.

A number $l \in \mathbb{R}$ is called a **lower bound** for A if for any $x \in A$, $x \geq l$.

↪ **Definition 1.5: The Least Upper Bound**

Let $A \subseteq \mathbb{R}$. A real number s is called the **least upper bound** for A if the following holds:

(a) s is an upper bound for A

(b) if b is any other upper bound for A, then $s \leq b$.

The least upper bound of a set A is *unique*, if it exists; if s and s' are two least upper bounds, then by (a), s and s' are upper bound for A, and by (b), $s \leq s'$ and $s' \leq s$, and thus $s = s'$.

This least upper bound is called the *supremum* of A, denoted $\sup(A)$.

↪ **Definition 1.6: The Greatest Lower Bound**

Let $A \subset \mathbb{R}$. A number $i \in \mathbb{R}$ is called the **greatest lower bound** for A if the following holds:

(a) i is a lower bound for A

(b) if l is any other lower bound for A, then $i \geq l$.

If i exists, it is called the *infimum* of A and is denoted $i = \inf(A)$, and is unique by the same argument used for $\sup(A)$.

↪ **Proposition 1.8**

Let[13] $A \subseteq \mathbb{R}$ and let s be an upper bound for A. Then $s = \sup(A)$ iff for any $\varepsilon > 0$, there exists $x \in A$ s.t. $s - \varepsilon < x$.

[13] Note that this, and proposition 1.9 that follows, are *not* definitions: they are restatements, and do technically require proof.

Proof. We have two statements:

I. $s = \sup(A)$;

II. For any $\varepsilon > 0$, $\exists x \in A$ s.t. $s - \varepsilon < x$;

and we desire to show that I $\iff$ II.

- I $\implies$ II: Let $\varepsilon > 0$. Then, since $s = \sup(A)$, $s - \varepsilon$ *cannot* be an upper bound for A (as s is the least upper bound, and thus $s - \varepsilon < s$ cannot be an upper bound at all). Thus, there exists $x \in A$ such that $s - \varepsilon < x$, and thus if I holds, II must hold.

- II $\implies$ I: suppose that this does not hold, ie II holds for an upper bound s for A, but $s \neq \sup(A)$. Then, there exists some upper bound b of A s.t. $b < s$. Take $\varepsilon = s - b$. $\varepsilon > 0$, and since II holds, there exists $x \in A$ such that $s - \varepsilon < x$. But since $s - \varepsilon = b$ and thus $b < x$, then b cannot be an upper bound for A, contradicting our initial condition. So, if II $\implies$ I does *not* hold, we have a "impossibility", ie a value b which is an upper bound for A which cannot be an upper bound, and thus II $\implies$ I.

■

↪ **Proposition 1.9:** ⋆

Let $A \subseteq \mathbb{R}$ and let i be a lower bound for A. Then $i = \inf(A) \iff$ for every $\varepsilon > 0$ there exists $x \in A$ s.t. $x < i + \varepsilon$.[14]

[14] Use similar argument to proof of previous proposition.

Remark 1.2. ?? *1.5 can also be expressed in terms of infimum. Define* $-A = \{-x : x \in A\}$. *Then, if* b *is an upper bound for* A*, then* $b \geq x \,\forall\, x \in A$*, then* $-b \leq -x \,\forall\, x \in A$*, ie -b is a lower bound of* $-A$*. Similarly, if* l *is a lower bound for* A*,* $-l$ *is an upper bound for* $-A$*.*

Thus, if A *is bounded from above, then*

$$-\sup(A) = \inf(-A),$$

and if A *is bounded from below,*

$$-\inf(A) = \sup(-A).$$

↪ **Axiom 1.6: AC (infimum)**

Let $A \subseteq \mathbb{R}$; if A bounded from below, $\inf(A)$ exists.

↪ **Definition 1.7:** max, min

Let $A \subseteq \mathbb{R}$. An $M \in A$ is called a *maximum* of A if for any $x \in A$, $x \leq M$. M is an upper bound for A, **but also** $M \in A$.

If M exists, then $M = \sup(A)$; M is an upper bound, and if b any other upper bound, then $b \geq M$, because $M \in A$, and thus $M = \sup(A)$.

NB: $M = \max(A)$ **need not** exist, while $\sup(A)$ must exist. Consider $A = [0, 1)$; $\sup(A) = 1$, but there exists no $\max(A)$.

The same logic exists for the existence of minimum vs infimum (consider $(0, 1)$, with no maximum nor minimum).

↪ **Theorem 1.3: Nested interval property of $\mathbb{R}$**

Let $I_n = [a_n, b_n] = \{x : a_n \leq x \leq b_n\}, n = 1, 2, 3\dots$ be an infinite sequence of bounded, closed intervals s.t.

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots I_n \supseteq I_{n+1} \supseteq \dots$$

Then, $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$ (note that this does *not* hold in $\mathbb{Q}$).

Proof. [15] We have $I_n = [a_n, b_n], I_{n+1} = [a_{n+1}, b_{n+1}], \ldots$. And the inclusion $I_n \supseteq I_{n+1}$. $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n, \forall n \geq 1$. So, the sequence a_n (left-end) is increasing, and the sequence b_n (right-end) is decreasing.

We also have that for any $n, k \geq 1$, $a_n \leq b_k$. We see this by considering two cases:

- Case 1: $n \leq k$, then $a_n \leq a_k$ (as a_n is increasing), and thus $a_n \leq a_k \leq b_k$.
- Case 2: $n > k$, then $a_n \leq b_n \leq b_k$ (again, as b_n is decreasing).

Let $A = \{a_n : n \in \mathbb{N}\}$. Then, A is bounded from above by *any* b_k (as in our inequality we showed above). Let $x = \sup(A)$, which must exist by **??** 1.5.

Note that as a result, $x \geq a_n$ for all n, and for all k, $x \leq b_k$, as x is the lowest upper bound and must be $\leq$ all other upper bounds, and so for all $n \geq 1$, $a_n \leq x \leq b_n$, ie $x \in I_n \, \forall n \geq 1$, and thus $x \in \bigcap_{n=1}^{\infty} I_n$ and so $\bigcap_{n=1}^{\infty} \neq \varnothing$. ■

[15] Sketch: show that the left-end points are increasing and the right-end points are decreasing. Show either that all the left-end points are bounded from above or that all the right-end points are bounded from below. As a result, there exists a sup/inf (depending on which end you choose) of the set of all the right/left points. For the sup case, all upper bounds must be $\geq$ sup, and thus the sup is in all I_n, and thus in their intersect, and thus the intersect is not empty.

Remark 1.3. *The proof above emphasized the left-end points; it can equivalently be proven via the right-end points, and using* $y = \inf(\{b_n : n \in \mathbb{N}\}) = \inf(B)$, *rather than* $\sup(A)$, *and showing that* $y \in \bigcap I_n$.

Remark 1.4 (⋆). *Note too that, if* $x = \sup(A)$ *and* $y = \inf(B)$, *then* $x, y \in \bigcap_{n=1}^{\infty} I_n$*; in fact,* $\bigcap_{n=1}^{\infty} I_n = [x, y]$. *This can be done by*

- *Use the main proof to show* $x \in \bigcap I_n$
- *Use the previous remark to show* $y \in \bigcap I_n$
- *Show* $x \leq y \implies [x, y] \subseteq \bigcap I_n$
- *Show* $\bigcap I_n \subseteq [x, y] \implies$ *equality.*

Remark 1.5. *The intervals* I_n must *be closed; if not, eg* $I_n = (0, \frac{1}{n})$, *then* $\bigcap_{n=1}^{\infty} I_n = \varnothing$. *Say* $\bigcap I_n \neq \varnothing$*; take then some* $x \in \bigcap I_n$. *Then,* $x \in (0, \frac{1}{n}) \, \forall n \in \mathbb{N}$. *But by proposition* 1.10, $\forall x \in \mathbb{R}, \exists N \in \mathbb{N}$ *s.t.* $\frac{1}{N} < x$. *Clearly,* x *must be greater than* 0 *to exist in the intersection; hence, there will always exist some sufficiently large* N *such that* $\frac{1}{N} < x \implies x \notin (1, \frac{1}{N}) \implies x \notin \bigcap I_n \implies \bigcap I_n = \varnothing$.

1.6 Density of Rationals in Reals

↪ **Proposition 1.10: Archimedean Property**

(a) For any $x \in \mathbb{R}$, there exists a natural number n s.t. $n > x$.

(b) For any $y \in \mathbb{R}$ satisfying $y > 0$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < y$.

Remark 1.6. *(a) states that $\mathbb{N}$ is not a bounded subset of $\mathbb{R}$.*

Remark 1.7. *(b) follows from (a) by taking $x = \frac{1}{y}$ in (a), then $\exists n \in \mathbb{N}$ s.t. $n > \frac{1}{y} \implies \frac{1}{n} < y$, and thus we need only prove (a).*

Remark 1.8. *Recall that $\mathbb{Q}$ is an ordered field (operations $+, \cdot$ and a relation $<$). $\mathbb{Q}$ can be extended to a larger ordered field with extended definitions of these operations/relations, such that it contains elements that are larger than any natural numbers (ie, not bounded above). This is impossible in $\mathbb{R}$ due to AC.*

Proof. Suppose (a) not true in $\mathbb{R}$, ie $\mathbb{N}$ is bounded from above in $\mathbb{R}$. Let $\alpha = \sup \mathbb{N}$, which exists by AC.

Consider $\alpha - 1$; since $\alpha - 1 < \alpha$, $\alpha - 1$ is not an upper bound of $\mathbb{N}$. So, there exists some $n \in \mathbb{N}$ s.t. $\alpha - 1 < n$; then, $\alpha < n + 1$ where $n + 1 \in \mathbb{N}$, and thus α is also not an upper bound, as there exists a natural number that is greater than α. This contradicts the assumption that $\alpha = \sup \mathbb{N}$, so (a) must be true. ■

↪ **Theorem 1.4: Density**

Let $a, b \in \mathbb{R}$ s.t. $a < b$. Then, $\exists x \in \mathbb{Q}$ s.t. $a < x < b$.

Remark 1.9. *If you take $a \in \mathbb{R}$ and $\varepsilon > 0$, then by the theorem, $\exists x \in \mathbb{Q}$ where $x \in (a - \varepsilon, a + \varepsilon)$. So any real number can be approximated arbitrarily closely (via choose of ε) by a rational number.*

Proof. Since $b - a > 0$, by (b) of proposition 1.10, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < b - a$, ie $na + 1 < nb$.

Let $m \in \mathbb{Z}$ s.t. $m - 1 \leq na < m$. Such an integer must exists since $\bigcup_{m \in \mathbb{Z}} [m-1, m) = \mathbb{R}$, the family $[m-1, m), m \in \mathbb{Z}$ makes partitions of $\mathbb{R}$. Then, $na < m$ gives that $a < \frac{m}{n}$. On the other hand, $m - 1 \leq na$ gives $m \leq na + 1 < nb$. So $\frac{m}{n} < b$ and it follows that $\frac{m}{n}$ satisfies $a < \frac{m}{n} < b$. ■

In the proof, we used the claim:

↪ **Proposition 1.11**

If $z \in \mathbb{R}$, then there exists $m \in \mathbb{Z}$ s.t. $m - 1 \leq z < m$.

Proof. Let S be a non-empty subset of $\mathbb{N}$. Then S has the least element; $\exists m \in S$ s.t. $m \leq n$, $\forall n \in S$.

We can assume $z \geq 0$; if $0 \leq z < 1$, then we are done (take $m = 1$), and assume that $z \geq 1$. Let now $S = \{n \in \mathbb{N} : z < n\}, \neq \varnothing$ by proposition 1.10, (a). Let m be the least element of S. It exists by Well-Ordering Property; then, since $m \in S$, $z < m$. But, we also have $m - 1 \leq z$, otherwise, if $z < m - 1$ then $m - 1 \in S$ and then m is not the least element of S. Thus, we have $m - 1 \leq z < m$, as required. ∎

↪ **Theorem 1.5**

The set J of irrationals is also dense in $\mathbb{R}$. That is, if $a, b \in \mathbb{R}, a < b$, $\exists$ irrational y s.t. $a < y < b$ (noting that $J = \mathbb{R} \setminus \mathbb{Q}$).

Proof. Fix $y_0 \in \mathbb{J}$. Consider $a - y_0$, $b - y_0$. $a - y_0 < b - y_0$, and by density of rationals, $\exists x \in \mathbb{Q}$ s.t. $a - y_0 < x < b - y_0$. Then, $a < y_0 + x < b$; let $y = x + y_0$, and we have $a < y < b$.

Note that y cannot be rational; if $y \in \mathbb{Q}$, $y = x + y_0 \implies y - x = y_0$, and since $x \in \mathbb{Q}$, $y - x \in \mathbb{Q} \implies y_0 \in \mathbb{Q}$, contradicting the original choice of $y_0 \notin \mathbb{Q}$. Thus, $y \in J$. ∎

↪ **Theorem 1.6**

$\exists$ a unique positive real number α s.t. $\alpha^2 = 2$.

Proof. We show both uniqueness, existence:[16]

Uniqueness: if $\alpha^2 = 2$ and $\beta^2 = 2, \alpha \geq 0, \beta \geq 0$, then $0 = \alpha^2 - \beta^2 = (\alpha - \beta)(\alpha + \beta) > 0$, and so $\alpha - \beta = 0 \implies \alpha = \beta$.

- Existence: consider the set $A = \{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 < 2\}$. A is not empty as $1 \in A$. The set of A is bounded above by 2, since if $x \geq 2$, then $x^2 \geq 4 > 2$, so $x \notin A$. So, by AC, $\sup A$ exists; let $\alpha = \sup A$. We will show that $\alpha^2 = 2$, by showing that both $\alpha^2 < 2$ and $\alpha^2 > 2$ are contradictions.

    $\alpha^2 < 2$

For any $n \in \mathbb{N}$ we expand

$$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} \leq \alpha^2 + \frac{2\alpha + 1}{n},$$

noting that $\frac{1}{n^2} \leq \frac{1}{n}$ for $n \geq 1$.

Let $y = \frac{2-\alpha^2}{2\alpha+1}$, which is strictly positive. By proposition 1.10, $\exists n_0 \in \mathbb{N}$ s.t.

$$\frac{1}{n_0} < \frac{2 - \alpha^2}{2\alpha + 1} \text{ or } \frac{2\alpha + 1}{n_0} < 2 - \alpha^2.$$

Substituting this n_0 into our inequality, we have

$$\left(\alpha + \frac{1}{n_0}\right)^2 \leq \alpha^2 + \frac{2\alpha + 1}{n_0} < \alpha^2 + 2 - \alpha^2 = 2.$$

Since $\alpha + \frac{1}{n_0}$ is positive, $\alpha + \frac{1}{n_0} \in A$. But, since $\alpha = \sup A$, $\alpha + \frac{1}{n_0} \leq \alpha$, which is impossible, so $\alpha^2 < 2$ cannot be true.

$\alpha^2 > 2$

Take $n \in \mathbb{N}$;

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}.$$

Now, let $y = \frac{\alpha^2-2}{2\alpha}$; $y > 0$, and by proposition 1.10, $\exists n_0 \in \mathbb{N}$ s.t.

$$\frac{1}{n_0} < \frac{\alpha^2 - 2}{2\alpha}, \text{ or } \frac{2\alpha}{n_0} < \alpha^2 - 2.$$

Substituting this n_0, we have

$$\left(\alpha - \frac{1}{n_0}\right)^2 > \alpha^2 - \frac{2\alpha}{n_0} > \alpha^2 + 2 - \alpha^2 = 2.$$

So for any $x \in A$, we have $\left(\alpha - \frac{1}{n_0}\right)^2 > 2 > x^2$. $\alpha - \frac{1}{n_0} > 0$, and $x > 0$, since $x \in A$. Then, $\left(\alpha - \frac{1}{n_0}\right)^2 > x^2$ gives that $\alpha - \frac{1}{n_0} > x$.

So, $\alpha - \frac{1}{n_0} > x$ for all $x \in A$. So $\alpha - \frac{1}{n_0}$ is an upper bound for A, but since $\alpha = \sup A$, $\alpha - \frac{1}{n_0} \geq \alpha$ ie $\alpha \geq \alpha + \frac{1}{n_0}$, which is impossible. So $\alpha^2 > 2$ cannot be true.

Thus, $\alpha^2 = 2$.

■

[16]Proof sketch: uniqueness is clear. Existence follows from showing that α^2 cannot be either $<$ or $>$ 2. This is done by contradiction, taking some number slightly

Remark 1.10. *A similar argument gives that for any $x \in \mathbb{R}$, $x \geq 0$, $\exists! \alpha \in \mathbb{R}$, $\alpha \geq 0$ such that $\alpha^2 = x$. This x is called the* square root *of x, denoted $\alpha = \sqrt{x}$.*

Remark 1.11. *For any natural number $m \geq 2$ and $x \geq 0$, $\exists! \alpha \in \mathbb{R}, \alpha \geq 0$ s.t. $\alpha^m = x$. The proof is similar, and we call α the m-th root of x.*

Remark 1.12. *Our last proof also gives that $\mathbb{Q}$ cannot satisfy AC. Suppose it does, ie any set in $\mathbb{Q}$ bounded from above has a supremum $\in \mathbb{Q}$. Then, consider $B = \{x \in \mathbb{Q} : x \geq 0$ and $x^2 < 2\}$; set $\alpha = \sup B$. The exact same proof can be used, but we will not be able to find an upper bound in $\mathbb{Q}$.*

1.7 Cardinality

↪ **Definition 1.8**

Let $f : A \to B$.

1. f injective (one-to-one) if $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$
2. f surjective (onto) if for any $b \in B \exists a \in A$ s.t. $f(a) = b$.
3. f bijective if both.

↪ **Definition 1.9: Composition**

If $f : A \to B, g : B \to C$, the *composite map* $h = g \circ f$ is define by $h(x) = g(f(x))$. Note that $h : A \to C$.

⊛ **Example 1.10**

Consider functions f, g.

1. If f, g injective, so is $h = g \circ f$
2. If f, g bijective, then so is h
3. If $\exists E \subseteq C$, then $h^{-1}(E) = f^{-1}(g^{-1}(E))$

↪ **Definition 1.10**

The inverse function[17] is defined only for bijective map $f : A \to B$. $y \in B$, $f^{-1}(y) = x$ where $x \in A$ s.t. $f(x) = y$.

[17] Not the same as the inverse *image* of a set by a function, which is defined for any function.

⊛ **Example 1.11**

1. $A = \mathbb{R}, B = (0, \infty), f(x) = e^x$. f is a bijection, and $f^{-1}(y) = \ln y, y \in (0, \infty)$.

2. $A = (-\frac{\pi}{2}, \frac{\pi}{2}, B = \mathbb{R})$. $f(x) = \tan x$, $f^{-1}(y) = \arctan y$

↪ **Definition 1.11: Equal Cardinalities**

Let A, B be two sets. We say A, B have the same cardinality, denote $A \sim B$ if there exists a bijective function $f : A \to B$.

⊛ **Example 1.12**

Let $E = \{2, 4, 6, \ldots\}$ (even natural numbers). Define $f : \mathbb{N} \to E$ by $f(n) = 2n$. Thus, f is a bijection, and $\mathbb{N} \sim E$.[18]

[18] See these independent notes for more.

↪ **Theorem 1.7**

The relation $\sim$ is a relation of equivalence.

1. $A \sim A$

2. if $A \sim B$, then $B \sim A$

3. if $A \sim B$ and $B \sim C$, then $A \sim C$

↪ **Definition 1.12: Countable**

A set A is *countable* if $\mathbb{N} \sim A$.

Remark 1.13. *According to this, finite sets are not countable; this is just a convention. Sometimes, we say a set is countable if it is finite or to above definition holds, where we say that a set is* countably infinite *if it is infinite and countable.*

Other times, finite sets are treated separately than countable sets.

↪ **Theorem 1.8**

Suppose that $A \subseteq B$.

1. If B is finite or countable, then so is A

2. If A is infinite and uncountable, then so is B

↪ **Definition 1.13: Cartesian Product**

If A, B sets, $A \times B = \{(a, b) : a, b \in A, B\}$.

↪ **Proposition 1.12**

$\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$; there exists a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

↪ **Proposition 1.13**

Let A be a set. The following are equivalent statements:

(a) A is finite or a countable set;

(b) there exists a surjection from $\mathbb{N}$ onto A;

(c) there exists a injection from A into $\mathbb{N}$.

Proof. We proceed by proving that each statement implies the next (and thus are equivalent).

- (a) $\implies$ (b): Suppose A is finite and has $\mathbb{N}$ elements. Then there exists a bijection $h : \{1, 2, \ldots n\} \to A$. We now define a map $f : \mathbb{N} \to A$, by setting

  $$f(m) = \begin{cases} h(m) & \text{if } m \leq n \\ h(n) & \text{if } m > n \end{cases}.$$

  f is surjective, and thus (b) holds. If (a) countable, $\exists$ bijection $f : \mathbb{N} \to A$, and any bijection is a surjection, so (b) also holds.

- (b) $\implies$ (c): Let $h : \mathbb{N} \to A$ be a surjection, whose existence is guaranteed by (b). Then, for any $a \in A$, the set

  $$h^{-1}(\{a\}) = \{m \in \mathbb{N} : h(m) = n\} \neq \varnothing,$$

  since h is a surjection. Then, by the well-ordering property of $\mathbb{N}$, the set $h^{-1}(\{a\})$ has a least element.
  If n is the least element of $h^{-1}(\{a\})$, we set $f(a) =$. This defines a function

  $$f : A \to \mathbb{N},$$

and we aim to show that f is injective, ie that $f(a_1) = f(a_2) \implies a_1 = a_2$.
Suppose $f(a_1) = f(a_2) = n$. Then, n is the least element of $h^{-1}(\{a_1\})$ and of $h^{-1}(\{a_2\})$, and in particular, $h(n) = a_1$ and $h(n) = a_2$, and thus $a_1 = a_2$ and so f is indeed injective.

- (c) $\implies$ (a): Let $f : A \to \mathbb{N}$ be an injection, whose existence is guaranteed by (c). Consider the range of f, ie

$$f(A) = \{f(a) : a \in A\}.$$

Since f an injection, f is a bijection between A and $f(A)$.
Otoh, $f(A) \subseteq \mathbb{N}$, and so by theorem 1.8, $f(A)$ is either finite or countable, and there exists a bijection between A and some set that is either fininte or countable. Thus, A must also be finite or countable, and so (a) holds.

■

↪ **Theorem 1.9**

Let $A_n, n = 1, 2, \ldots$ be a sequence of sets such that each A_n is either finite or countable. Then, their union

$$A = \bigcup_{n=1}^{\infty} A_n$$

is also either finite or countable.

Proof. We will use (a) $\iff$ (b) from proposition 1.13 to prove this.

Since each A_n finite or countable, by (a) $\implies$ (b), there exists a surjection

$$\varphi_n : \mathbb{N} \to A_n.$$

Now, let $h : \mathbb{N} \times \mathbb{N} \to A$, (the union) by setting

$$h(n, m) = \varphi_n(m).$$

We aim to show that h is also surjective.
If $a \in \bigcup_{n=1}^{\infty} A_n$, then $a \in A_n$ for some $n \in \mathbb{N}$. Since $\varphi_n : \mathbb{N} \to A_n$ is a surjection, there exists an $m \in \mathbb{N}$ s.t. $\varphi_n(m) = a$. By definition of h, we have

$$h(n, m) = a,$$

and thus h is a surjection.

By proposition 1.12, there exists a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, and we can define the composite map

$$h \circ f : \mathbb{N} \to A \, (= \cup_{n=1}^{\infty} A_n),$$

which is a surjection as both h, f are surjections. So, there exists a surjection from $\mathbb{N} \to A$, and by proposition 1.13, (b) $\implies$ (a), and thus $A = \bigcup_{n=1}^{\infty} A_n$ is also finite or countable.

■

Remark 1.14. *If $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is either finite or countable, and at least one A_n is countable, then A is countable.*

Remark 1.15. *If $A_1, \ldots, A_n$ are finitely many finite or countable sets then their union $A_1 \cup \cdots \cup A_n$ is also finite or countable (essentially just previous proof where we use n instead of ∞ for the upper limit of the union...).*

↪ **Theorem 1.10**

The set $\mathbb{Q}$ of rational numbers is countable.

Proof. We write

$$\mathbb{Q} = A_0 \cup A_1 \cup A_2,$$

where $A_0 = \{0\}$, $A_1 = \{\frac{m}{n} : m, n \in \mathbb{N}\}$, and $A_2 = \{-\frac{m}{n} : m, n \in \mathbb{N}\}$.
Let us show that A_1 is countable; define

$$h : \mathbb{N} \times \mathbb{N} \to A_1, f(m, n) = \frac{m}{n}.$$

h is clearly a surjection; if $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection, then by proposition 1.12, $h \circ f : \mathbb{N} \to A_1$ is a surjection. By proposition 1.13, A_1 is countable.
We prove that A_2 countable in essentially the same way.
Then, $A_0 \cup A_1 \cup A_2$ is also countable, as it is the union of countable sets, and thus $\mathbb{Q}$ is also countable. ■

↪ **Theorem 1.11**

The set $\mathbb{R}$ of real numbers is uncountable.[19]

Proof. We will argue by contradiction; suppose $\mathbb{R}$ is countable, then show that the nested interval property (theorem 1.3) of the real line fails.

Let $f : \mathbb{N} \to \mathbb{R}$ be a bijection, setting $f(1) = x_1, f(2) = x_2, \ldots, f(n) = x_n, \ldots$; we can then list the elements of $\mathbb{R}$ as $\mathbb{R} = \{x_1, x_2, x_3, \ldots, x_n, \ldots\}$.

We can now construct a sequence $I_n, n \in \mathbb{N}$ of bounded, closed intervals, such that I_1 does not contain x_1.

If $x_2 \notin I_1$, then $I_2 = I_1$. If $x_2 \in I_1$, then divide I_1 into four equal closed intervals.

Call the leftmost/rightmost of these intervals I_1' and I_1'' respectively. We know that $x_2 \in I_1$, so we must have that either $x_2 \notin I_1'$ or $x_2 \notin I_1''$ If $x_2 \notin I_1'$, then $I_2 = I_1'$. If $x_2 \notin I_1''$, then $I_2 = I_1''$.

Thus, we have constructed I_1, I_2 s.t.

$$I_1 \supseteq I_2 \text{ and } x_1 \notin I_1, x_2 \notin I_2.$$

Consider x_3; if $x_3 \notin I_2$, then $I_3 = I_2$. If $x_3 \in I_2$, we repeat the "dividing" process as before. Since $x_3 \in I_2$, either $x_3 \notin I_2'$ or $x_3 \notin I_2''$. If $x_3 \notin I_2'$, $I_3 = I_2'$. Else, if $x_3 \notin I_2''$, $I_3 = I_2''$.

We have now that

$$I_1 \supseteq I_2 \supseteq I_3 \text{ and } x_1 \notin I_1, x_2 \notin I_2, x_3 \notin I_3,$$

and we can continue this construction to obtain an infinite sequence of bounded, closed intervals I_n s.t.

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots,$$

and for each n, $x_n \notin I_n$.

Consider the intersection of all these I_n's,

$$\bigcap_{n=1}^{\infty} I_n.$$

For every m, $x_m \notin I_m$, so for every $m \in \mathbb{N}$, $x_m \notin \bigcap_{n=1}^{\infty} I_n$, and so $\mathbb{R} = \{x_1, x_2, \ldots x_m, \ldots\}$ has an empty intersection with this intersection, ie

$$\mathbb{R} \cap \left(\bigcap_{n=1}^{\infty} I_n \right) = \varnothing.$$

Otoh, $\bigcap_{n=1}^{\infty} I_n \subseteq \mathbb{R}$, so we must have that $\bigcap_{n=1}^{\infty} I_n = \varnothing$ contradicting the nested interval

[19] Proof sketch: by contradiction. Assume that a bijection exists, and show that it cannot be a surjection by the previous props/thms. Specifically, carefully construct nested intervals I_n, for which $x_i \notin I_i$, and then show that the intersection of all these intervals is empty, contradicting the nested interval property of the real line.
See pg. 25 of Abbott's Analysis for a more concise proof in the same language.

property of the real line which states that this intersection must not be empty. We thus have a contradiction, and our assumption that $\mathbb{R}$ countable fails. [20] ■

[20] Note that theorem 1.3 is built upon the Axiom of Completeness, a "fact" of $\mathbb{R}$ (what makes it "distinct" from $\mathbb{Q}$, $\mathbb{N}$, etc). Thus, we are really just using AC, with some abstractions sts.

↪ **Proposition 1.14**

The set J of all irrational numbers in $\mathbb{R}$ is uncountable.

Proof. We have that $\mathbb{R} = \mathbb{Q} \cup J$. If J countable, then $\mathbb{R}$ would also be countable as the union of two countable sets (as we showed $\mathbb{Q}$ countable in theorem 1.10). $\mathbb{R}$ uncountable, so J is also uncountable. ■

↪ **Proposition 1.15**

The set $(-1, 1) \subseteq \mathbb{R}$ is uncountable.

Proof. We can write $\mathbb{R} = \bigcup_{n=1}^{\infty}(-n, n)$. If each $(-n, n)$ is countable, then $\mathbb{R}$ would also be countable, as a countable union of countable sets. Thus, there must exist some $n_0 \in \mathbb{N}$ s.t. $(-n_0, n_0)$ is not countable. The map

$$f : (-n_0, n_0) \to (-1, 1), f(x) = \frac{x}{n_0}$$

is a bijection, and so $(-1, 1)$ is uncountable. ■

⊛ **Example 1.13**

Show that the map

$$f(x) = \frac{x}{1 - x^2}$$

is a bijection between $(-1, 1)$ and $\mathbb{R}$ ie $(-1, 1) \sim \mathbb{R}$.

Proof. Surjection is fairly trivial (if stuck, consider the graph of the function).
Injection; given $f(x) = f(y)$ where $x, y \in (-1, 1)$,

$$\begin{aligned} \frac{x}{1-x^2} &= \frac{y}{1-y^2} \\ x - xy^2 &= y - yx^2 \\ x - y = xy^2 - yx^2 &= xy(y - x) \\ x - y &= -xy(x - y) \\ \implies -xy = 1 \implies xy &= -1, \text{ or } x - y = 0 \end{aligned}$$

$xy = -1$ is impossible given the domain of the function, hence $x - y = 0 \implies$

$x = y$, as desired. ∎

↪ Proposition 1.16

Any bounded non-empty open interval $(a, b) \in \mathbb{R}$ is uncountable.

Proof. We will construct a bijection $f : (a, b) \to \mathbb{R}$ so that $(a, b) \sim \mathbb{R}$. Since $\mathbb{R}$ is uncountable, so must (a, b).

The map

$$f(x) = \frac{2(x-a)}{b-a} - 1$$

is a bijection between (a, b) and $(-1, 1)$, and we have shown that $(-1, 1) \sim \mathbb{R}$, so $(a, b) \sim \mathbb{R}$, and thus any open interval has the same cardinality as $\mathbb{R}$. ∎

⊛ Example 1.14

Prove that $\exists$ bijection between $[0, 1)$ and $(0, 1)$, and conclude that $[0, 1) \sim (0, 1) \sim \mathbb{R}$. Then conclude for any $a < b$, $[a, b) \sim \mathbb{R}$.

1.7.1 Power Sets

↪ Definition 1.14: Power Set

Let A be a set. The *power set* of Am denoted $\mathcal{P}(A)$ is the collection of all subsets of A.

Generally, if A finite of size n, $\mathcal{P}(A)$ has 2^n elements.

↪ Theorem 1.12: Cantor Power Set Theorem

Let A be any set. Then there exists no surjection from A onto $\mathcal{P}(A)$. [21]

[21] Certified Classic

Proof. Suppose that there exists a surjection,

$$f : A \to \mathcal{P}(A).$$

Let $D \subseteq A$ defined as

$$D = \{a \in A : a \notin f(a)\}.$$

Since $D \subseteq \mathcal{P}(A)$, and f is surjective, there must exist some $a_0 \in A$ s.t. $f(a_0) = D$. We have two cases:

1. $a_0 \in D$. But then, by definition of D, $a_0 \notin f(a_0) = D$, so $a_0 \in D$ is not possible as it implies $a_0 \notin D$.

2. $a_0 \notin D$. But then, since $D = f(a_0)$, $a_0 \notin f(a_0)$, and so by definition of D, $a_0 \in D$, which is again not possible.

So, the assumption of a surjection existing has led to $a_0 \in A$ such that neither $a_0 \in D$ nor $a_0 \notin D$, which is impossible. Thus there can be no surjective f.
Notice, though, that there exists an injection $A \to \mathcal{P}(A), a \mapsto \{a\}$, and thus there is an injection but no bijection.
Thus, we can say that $\mathcal{P}(A)$ is strictly bigger than A.

■

2 Sequences

2.1 Definitions

↪ **Definition 2.1**

Let A be a set. An A-valued sequence indexed by $\mathbb{R}$ is a map

$$x : \mathbb{N} \to A.$$

The value $x(n)$ is called the n-th element of the sequence. One writes $x(n) = x_n$, or lists its elements

$$\{x_1, x_2, x_3, \dots\} \equiv \{x_n\}_{n\in\mathbb{N}} \equiv (x_n)_{n\in\mathbb{N}} \equiv \{x_n\}.$$

↪ **Definition 2.2: Convergence**

We say that a sequence (x_n) converges to a real number x if for every $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. for all $n \geq N$ we have

$$|x_n - x| < \varepsilon.$$

If sequence (x_n) converges to x, we write $\lim_{n\to\infty} x_n = x$.

⊛ **Example 2.1**

Let (x_n) be a sequence defined by $x_n = \frac{1}{n}, n \in \mathbb{N}$, then $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\varepsilon > 0$. Let $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$. Then for $n \geq N$, we have that

$$0 < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

So, for $n \geq N$, $|x_n - 0| < \varepsilon$, and so the limit is 0. ■

↪ Definition 2.3: Quantifier of Limit ⋆

The limit can be written in terms of quantifiers.

$$\lim_{n\to\infty} x_n = x$$

means that

$$(\forall\,\varepsilon > 0)(\exists N \in \mathbb{N})(\forall\,n \geq N)(|x_n - x| < \varepsilon).$$

⊛ Example 2.2

Prove[22] that

$$\lim_{n\to\infty} \frac{n^2+1}{n^2} = 1.$$

Proof. Let $\varepsilon > 0$. Let N be a natural number such that $N > \frac{1}{\sqrt{\varepsilon}}$. Then, for $n \geq N$,

$$\left|\frac{n^2+1}{n^2} - 1\right| = \left|\frac{n^2+1-n^2}{n^2}\right| = \frac{1}{n^2} \leq \frac{1}{N^2} < \varepsilon.$$

■

[22] Work backwards to start; how can you simply the sequence (that is, build a string of inequalities) such that defining an N in terms of ε becomes apparent?

↪ Definition 2.4: Divergent Sequences

If a sequence (x_n) does not converge to any real number x, we say that the sequence is divergent. For instance, consider

$$x_n = (-1)^n, n \geq 1.$$

The sequence alternates between 1 and -1 and so intuitively does not converge. How do we prove it?

Proof. By contradiction; suppose that $x_n = (-1)^n$ be a converging sequence. Let $x = \lim_{n\to\infty} x_n$. Take $\varepsilon = 1$, then $\exists N \in \mathbb{N}$ s.t. for all $n \geq N$ we have that $|x - x_n| < \varepsilon = 1$.

Consider indices $n = N, n = N + 1$. We have

$$|x_{N+1} - x_N| = |x_{n+1} - x + x - x_N| \le \underbrace{|x_{N+1} - x| + |x - x_N|}_{\text{triangle inequality}} < 1 + 1 = 2.$$

But we also have that

$$|(-1)^{N+1} - (-1)^N| = |(-1)^{N+1} + (-1)^{N+1}| = 2,$$

We thus have that $2 < 2$, which is a contradiction. Thus, x_n is not convergent. ■

⊛ **Example 2.3**

Evaluate the following examples using the ε definition:

1. $\lim_{n\to\infty} \frac{\sin n}{\sqrt[3]{n}} = 0$
2. $\lim_{n\to\infty} \frac{n!}{n^n} = 0$
3. $\lim_{n\to\infty} \frac{(1+2+\cdots+n)^2}{n^4} = \frac{1}{4}$

Proof. 1. For all $\varepsilon > 0$; take $\frac{1}{N} < \varepsilon^3 \implies \frac{1}{\sqrt[3]{N}} < \varepsilon$. Then, $\forall\, n \ge N$,

$$n \ge N \implies \sqrt[3]{n} \ge \sqrt[3]{N} \implies \frac{1}{\sqrt[3]{n}} \le \frac{1}{\sqrt[3]{N}}$$

$$-1 \le \sin n \le 1 \implies |\sin n| \le 1 \implies \left|\frac{\sin n}{\sqrt[3]{n}}\right| \le \left|\frac{1}{\sqrt[3]{N}}\right| \le \frac{1}{\sqrt[3]{N}} < \varepsilon$$

$$\implies \lim_{n\to\infty} \frac{\sin n}{\sqrt[3]{n}} = 0$$

2. Take $\frac{1}{N} \le \varepsilon$. Then, $\forall\, \varepsilon > 0$, $\forall\, n \ge N \implies \frac{1}{n} \le \frac{1}{N}$,

$$\frac{n!}{n^n} > 0 \implies \left|\frac{n!}{n^n}\right| = \frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots 1}{n \cdot n \cdots n} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \frac{n-2}{n} \cdots \frac{1}{n}$$

$$\le 1 \cdot 1 \cdots 1 \cdot \frac{1}{n}$$

$$\le \frac{1}{n} \le \frac{1}{N} < \varepsilon$$

$$\implies \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

3. Note first that $(1 + 2 + \cdots + n)^2 = (\frac{n(n+1)}{2})^2$ (see example 1.1). Take $\frac{1}{N} < \frac{\varepsilon}{2}$;

then, $\forall\, \varepsilon > 0$, we have that $\forall\, n \geq N$,

$$\begin{aligned}\left|\frac{(1+2+\cdots+n)^2}{n^4} - \frac{1}{4}\right| &= \frac{\frac{n^2(n+1)^2}{4}}{n^4} - \frac{n^4}{n^4} = \frac{n^4+2n^3+n^2-n^4}{n^4} \\ &= \frac{2n^3+n^2}{n^4} = \frac{2n+1}{n^2} \leq \frac{2n}{n^2} \leq \frac{2}{n} \leq \frac{2}{N} < \varepsilon \\ &\implies \lim_{n\to\infty} \frac{(1+2+\cdots+n)^2}{n^4} = \frac{1}{4}\end{aligned}$$

■

2.2 Properties of Limits

↪ Lemma 2.1: Triangle Inequality

For $x, y, z \in \mathbb{R}$,

$$(i) \quad |x+y| \leq |x| + |y|; \qquad (ii) \quad |x-y| \leq |x-z| + |z-y|$$ [23]

[23] Generally, proofs involving limits will consist of 1) picking/defining an ε based on given limit/series definitions, and then 2) using triangle inequality/related techniques to reach the desired conclusion.

Sketch proof. (i): $|x+y| = \begin{cases} x+y & x+y \geq 0 \\ -(x+y) & x+y \leq 0 \end{cases}$. So if $x+y \geq 0$, $|x+y| = x+y \leq |x|+|y|$.

If $x+y > 0$, $|x+y| = -(x+y) = (-x)+(-y) \leq |x|+|y|$.

(ii): $|x-y| = |x-z+z-y| \leq |x-z| + |z-y|$ (using (i)). ■

↪ Theorem 2.1: ⋆

A limit of a sequence is unique. In other words, if the sequence is converging, then its limit is unique. The sequence cannot converge to two distinct numbers x and y.[24]

[24] Proof sketch: contradiction, assume two distinct limits, and take ε as their midpoint. Arrive at a contradiction by using triangle inequalities to show that $|x-y| < |x-y|$, and thus the limits cannot be distinct.

Proof. By contradiction; suppose $\exists (x_n)$ s.t. $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} x_n = y$, and that $x \neq y$.

Take $\varepsilon = \frac{|x-y|}{2}$. Since $x \neq y$, we have that $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, $\exists N_1 \in \mathbb{N}$ s.t. for $n \geq N_1$, $|x_n - x| < \varepsilon$.

Similarly, since $\lim x_n = y$, $\exists N_2 \in \mathbb{N}$ s.t for $g \geq N_2$, $|x_n - y| < \varepsilon$.

Take some $n \geq \max(N_1, N_2)$; then

$$|x - y| = |x - x_n + x_n - y| \leq |x - x_n| + |x_n - y|$$
$$< \varepsilon + \varepsilon = |x - y|$$
$$\implies |x - y| < |x - y|, \perp$$

■

↪ Theorem 2.2

Any converging sequence is bounded.[25]

In other words, if (x_n) is a converging sequence,

$$\exists M > 0 \text{ s.t. } |x_n| \leq M \,\forall\, n \geq 1.$$

Proof. Let (x_n) be a converging sequence, and $x = \lim_{n \to \infty} x_n$. Take $\varepsilon = 1$ in the definition of the limit; then, $\exists N \in \mathbb{N}$ s.t. $\forall\, n \geq N$, $|x_n - x| < 1$.

This gives that for $n \geq N$, $|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$.

Let now $M = |x_1| + |x_2| + \cdots + |x_{N-1}| + (1 + |x|)$. Then, for any $n \geq 1$, $|x_n| \leq M$;

If $n \leq N - 1$, then $|x_n|$ is a summand in M, and thus $|x_n| \leq M$.

If $n \geq N$, then we have by the choice of N that $|x_n| < 1 + |x| \leq M$.

Thus, for all $n \geq 1$, $|x_n| \leq M$, and is thus bounded given (x_n) converges. ■

[25] Take $\varepsilon = 1$, which is greater than $|x_n - x|$ by limit definition for $n \geq N$ for some N. We then use this to show that $|x_n| < 1 + |x|$, then construct a summation M such that it bounds $|x_n|$; it is equal to $|x_1| + |x_2| + \cdots$ up to $|x_{N-1}|$, then plus $1 + |x|$. We have finished.

↪ Proposition 2.1: Algebraic Properties of Limits

Let $(x_n), (y_n)$ be sequences such that[26]

$$\lim x_n = x, \quad \lim y_n = y.$$

Then:

1. For any constant c, $\lim c \cdot x_n = c \cdot \lim x_n = c \cdot x$
2. $\lim(x_n + y_n) = \lim x_n + \lim y_n = x + y$
3. $\lim x_n \cdot y_n = (\lim x_n)(\lim y_n) = x \cdot y$
4. Suppose $y \neq 0$, $y_n \neq 0 \,\forall\, n \geq 1$. Then, $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n} = \frac{x}{y}$

[26] Note that the contrary of these statements need not hold; ie, if $\lim(x_n \cdot y_n)$ exists, this does not imply the existence of $\lim x_n$ and $\lim y_n$. Consider example 2.4

Remark 2.1. *Let X be the collection of all sequences of real numbers, $X = \{(x_n) : x_n$ is a sequence$\}$. If $(x_n) \in X$ and $c \in \mathbb{R}$, we can define $c \cdot (x_n) = (c \cdot x_n)$[27]; this defines* scalar multiplication *on X.*
We can also define addition*; if (x_n) and (y_n) are two sequences in X, then $(x_n) + (y_n) = (x_n + y_n)$. Then, with these two operations X is a* vector space.

[27]NB: this denotes c multiplying to each nth element in x_n, ie $c \cdot x_1, c \cdot x_2$, etc

⊛ **Example 2.4**

Take $x_n = (-1)^n, y_n = (-1)^{n+1}, n \geq 1$.
$(x_n) + (y_n) = 0, x_n \cdot y_n = -1$, and so $\lim x_n + y_n = 0, \lim x_n \cdot y_n = -1$, while neither $\lim x_n$ nor $\lim y_n$ exist.

Proof (part 3. of proposition 2.1). Take[28] $\lim x_n = x, \lim y_n = y$. Since (x_n) is converging, it is bound by theorem 2.2, and there exists $M > 0$ s.t. $\forall\, n \geq 1, |x_n| \leq M$.
Now,

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n y_n - x_n y| + |x_n y - xy| \\ &= |x_n| \cdot |y_n - y| + |y| \cdot |x_n - x| \\ &\leq M \cdot |y_n - y| + |y| \cdot |x_n - x| \quad (i) \end{aligned}$$

Let $\varepsilon > 0$; since $\lim y_n = y$, there exists $N_1 \in \mathbb{N}$ s.t. $n \geq N_1, |y_n - y| < \frac{\varepsilon}{2M}$. Sim, since $\lim x_n = x, \exists N_2 \in \mathbb{N}$ s.t. $|x_n - x| < \frac{\varepsilon}{2(|y|+1)}$
Let $N = \max(N_1, N_2), n \geq N$. Then, we have, with (i),

$$\begin{aligned} (i) \quad |x_n y_n - xy| &\leq M \cdot |y_n - y| + |y| \cdot |x_n| - x \\ &< M \cdot \frac{\varepsilon}{2M} + |y| \cdot \frac{\varepsilon}{2(|y|+1)} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2}. \end{aligned}$$

Thus, for $n \geq N, |x_n y_n - xy| < \varepsilon$, and by definition of the limit, $\lim x_n y_n = xy$. ■

[28]Proof sketch: take an upper bound of x_n. Then, show that $|x_n y_n - xy| < \varepsilon$, by using triangle inequalities to show inequality to a combination of M, arbitrarily small values (by def of limits of x_n, y_n resp,), and $|y|$.

↪ **Theorem 2.3: Order Properties of Limits**

Let $(x_n), (y_n)$ be two sequences such that

$$\lim x_n = x, \quad \lim y_n = y.$$

1. $x_n \geq 0 \,\forall\, n \implies x \geq 0$.

2. $x_n \geq y_n \, \forall \, n \implies x \geq y$.

3. c is constant since $c \leq x_n \, \forall \, n \geq 1 \implies c \leq x$. $x_n \leq c \, \forall \, n \geq 1 \implies x_n \leq c$.

Remark 2.2. *2., 3. follow from 1. Set* $z_n = x_n - y_n \, \forall \, n \geq 1$. *Then,* $z_n \geq 0 \, \forall \, b \geq 1$, $\lim z_n = \lim(x_n - y_n) = \lim x_n - \lim y_n$ *(as these limits exist)* $= x - y$. *By 1.,* $\lim z_n \geq 0$, *and so either* $x - y \geq 0$ *or* $x \geq y$.

Proof of 1. Suppose 1. does not hold; suppose $\exists (x_n)$ s.t. $\lim x_n = x$, $x_n \geq 0 \, \forall \, \geq$, but $x < 0$.

Let $\varepsilon > 0$ s.t. $x < -2\varepsilon < 0$. With this ε, $\lim x_n = x$ gives that $\exists N \in \mathbb{N}$ s.t. $\forall \, n \geq N$, $|x_n - x| < \varepsilon$, or particularly, $x_n - x < \varepsilon$.

Then, $x_n < \varepsilon + x$, and since $x < -2\varepsilon$, we have $\forall \, n \geq N$, $x_n < -\varepsilon$, and thus $\forall \, n \geq N$, $x_n < 0$, a contradiction. ■

↪ **Theorem 2.4: The Squeeze Theorem**

Let $(x_n), (y_n), (z_n)$ be sequences such that $x_n \leq y_n \leq z_n$, $\forall \, n \geq 1$, and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = \ell$, then $\lim_{n\to\infty} y_n = \ell$.[29]

[29] Sketch: This follows a similar technique to many that follow. Use the definitions of the limits of x_n, z_n to take an arbitrary ε, and an N for each respective limit. Take the max of these N's, and show that for all $n \geq \max N_i$, you can show that f $y_n - l$ is less than ε and greater than $-\varepsilon$. Really, this is just a proof of applying definitions correctly.

Proof. Let $\varepsilon > 0$. Since $\lim x_n = \ell$, there $\exists N_1 \in \mathbb{N}$ s.t. $\forall \, n \geq N_1$, $|x_n - \ell| < \varepsilon$.
Since $\lim z_n = \ell$, there $\exists N_2 \in \mathbb{N}$ s.t. $\forall \, n \geq N_2$, $|z_n - \ell| < \varepsilon$.
Take $N = \max\{N_1, N_2\}$ and take $n \geq N$. Then,

$$y_n \leq z_n \implies y_n - \ell \leq z_n - \ell \leq |z_n - \ell| < \varepsilon,$$

since $n \geq \max\{N_1, N_2\} \implies n \geq N_2$.
Now, we have that

$$y_n \geq x_n \implies y_n - \ell \geq x_n - \ell > -\varepsilon,$$

since $|x_n - \ell| < \varepsilon$ for $n \geq N_1$, and our n is $\geq \max\{N_1, N_2\}$. Thus, for $n \geq N$,

$$-\varepsilon < y_n - \ell < \varepsilon \implies |y_n - \ell| < \varepsilon,$$

and thus $\lim y_n = \ell$, by definition. ■

↪ **Definition 2.5: Increasing/Decreasing**

A sequence (x_n) is called *increasing* if $x_{n+1} \geq x_n \, \forall \, n \in \mathbb{N}$, and is *decreasing* if $x_{n+1} \leq x_n \, \forall \, n \in \mathbb{N}$.

↪ **Definition 2.6: Bounded from above/below**

A sequence (x_n) is called *bounded* from above if there exists some $M \in \mathbb{R}$ s.t. $x_n \leq M \, \forall \, n \geq 1$.

Sequence (x_n) is bounded from below if there exists some $M \in \mathbb{R}$ s.t. $x_n \geq M \, \forall \, n \geq 1$.

↪ **Theorem 2.5: Monotone Convergence Theorem**

The following relate to bounded above/below and increasing/decreasing sequences:[30]

1. Let (x_n) be an increasing sequence that is bounded from above. Then (x_n) is converging.
2. Let (x_n) be a decreasing sequence that is bounded from below. then (x_n) is converging.

[30] Sketch: 1,2 are proven very similarly. For 1., take the set of all x_n in the given sequence. Since the sequence is bounded, then so is the set, and so we can take its supremum. Use the ε definition of sup to show that this supremum is also the limit of the sequence (basically, a bunch of inequalities, and being careful with definitions). 2. follows identically but using the infimum.

Proof (of 1). Let $A = \{x_n : n \geq 1\}$. Since (x_n) is bounded above by M, the set A is bounded from above. Let $\alpha = \sup A$, which exists by AC.

Let $\varepsilon > 0$. Since α is the least upper bound for A, $\alpha - \varepsilon$ is *not* an upper bound of A ($\alpha - \varepsilon < \alpha$). Hence, there must exist some $N \in \mathbb{N}$ such that $\alpha - \varepsilon < x_N$ (if it didn't exist, then α wouldn't be the supremum …). Then, for $n \geq N$, and since (x_n) increasing,

$$\alpha - \varepsilon < x_N \leq x_n \leq \alpha.$$

Then, for all $n \geq N$,

$$\alpha - \varepsilon < x_n \leq \alpha \implies -\varepsilon < x_n - \alpha \leq 0,$$

and so $|x_n - \alpha| < \varepsilon$ for $n \geq N$. By definition, $\alpha = \lim x_n$. ■

⊛ **Example 2.5**

A sequence (x_n) is called *eventually increasing* if there exists some $N_0 \in \mathbb{N}$ s.t. $\forall \, n \geq N_0, x_{n+1} \geq x_n$. If (x_n) is eventually increasing and bounded from above, $\lim x_n = \alpha$ exists.

Proof. (Sketch) If (x_n) eventually increasing, say $\forall \, n \geq N_0$, and bounded above, then if we consider x_n' as the sequence of x_n where $n' = n - N_0$, it must converge by Monotone Convergence Theorem. Hence, taking $N = N_0$, x_n too must converge.

■

Example 2.6

Let (x_n) be a sequence defined recursively by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2 + x_n}, n \geq 1$. So $x_2 = \sqrt{2+\sqrt{2}}, x_3 = \sqrt{2+\sqrt{2+\sqrt{2}}} \cdots, x_n = 2\cos\frac{\pi}{2^{n+1}}, n \geq 1$. Show that $\lim x_n = 2$.

Proof. We will prove this using the Monotone Convergence Thm by showing that the x_n is bounded from above and increasing, which implies that the limit exists. We will then find the actual limit.

Recall that $n \geq 1, x_n \leq 2$. We will prove this by induction. Let $S \subseteq \mathbb{N}$ be the set of indices such that $x_n \leq 2$. Since $x_1 = \sqrt{2} < 2, 1 \in S$. Now suppose some $n \in S$, ie $x_n \leq 2$. Then, we have that $x_{n+1} = \sqrt{2 + x_n} \leq \sqrt{2+2} = 2 \implies x_{n+1} \leq 2$. Thus, by induction, $n \in S \implies n+1 \in S \implies S = \mathbb{N}$, ie $x_n \leq 2 \,\forall\, n \in \mathbb{N}$. Thus, our sequence is bounded from above.

We now prove that (x_n) is increasing. Let $S \subseteq \mathbb{N}$ s.t. $n \in S \iff x_{n+1} \leq x_n$. $x_2 = \sqrt{2+\sqrt{2}} \geq \sqrt{2} = x_1 \implies x_1 \leq x_2 \implies 1 \in S$. Suppose $n \in S \implies x_{n+1} \geq x_n$. Then, $x_{n+2} = \sqrt{2 + x_{n+1}} \geq \sqrt{2 + x_n} = x_{n+1} \implies n+1 \in S$. Thus, $S = \mathbb{N}$, so $x_{n+1} \geq x_n \,\forall\, n \in \mathbb{N}$.

So the sequence (x_n) is increasing and bounded from above, and thus $\exists \lim x_n = \alpha$. To find the value of α, consider $x_{n+1} = \sqrt{2 + x_n}$, or $x_{n+1}^2 = 2 + x_n$. We can also write that $\alpha = \lim x_n = \lim x_{n+1}$.[31] We then have that $\lim x_{n+1} = \alpha \implies \lim x_{n+1}^2 = \alpha^2$, and thus $x_{n+1}^2 = 2 + x_n \implies \lim x_{n+1}^2 = \lim(2 + x_n) \implies \alpha^2 = 2 + \alpha \implies \alpha = 2, -1$. $x_n \geq 0 \,\forall\, n$, by Order Limit Theorem, and so $\alpha \geq 0$ and thus $\alpha = 2$. ■

[31] Add proof

Corollary 2.1

For $a, b > 0$, then $\frac{1}{2}(a+b) \geq \sqrt{ab}$

Proof. $\left[\frac{1}{2}(a+b)\right]^2 = \frac{1}{4}(a^2 + 2ab + b^2) \geq ab \implies \frac{1}{2}(a+b) \geq \sqrt{ab}$ ■

Example 2.7

Let (x_n) be defined recursively by $x_1 = 2$ and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$ for $n \geq 1$. Then, (x_n) is converging and $\lim x_n = \sqrt{2}$.

Proof. We[32] will show that (x_n) bounded from below and decreasing, implying the limit exists. We will show that for all n, $x_n \geq \sqrt{2}$. For $n = 1$, $2 \geq \sqrt{2}$. For $n > 1$, we will use corollary 2.1. We then have that $x_{n+1} = \frac{1}{2}(x_n + \frac{2}{x_n}) \geq \cdots \geq \sqrt{2} \implies x_n \geq \sqrt{2} \, \forall \, n \geq 1$, ie, it is bounded from below.

We will now show that the sequence is decreasing.

$$x_n - x_{n+1} = x_n - \frac{1}{2}(x_n + \frac{2}{x_n}) = \frac{1}{2}x_n - \frac{1}{x_n} = \frac{1}{2x_n}(x_n^2 - 2) \geq \frac{1}{2x_n}(\sqrt{2}^2 - 2) \geq 0,$$

where the second-to-last inequality holds from the lower bound we found on x_n. Having shown that x_n decreasing and is bounded from below, we conclude that it converges by Monotone Convergence Theorem. To find its limit, let $L := \lim x_n$. Then,

$$\lim x_n = \lim \left(\frac{1}{2}x_{n-1} + \frac{2}{x_{n-1}}\right) = \frac{1}{2}\lim x_{n-1} + \lim \frac{1}{x_{n-1}},$$

and since the limit of x_n is equal to the limit of x_{n-1}, we have that

$$L = \frac{1}{2}L + \frac{1}{L} \implies L^2 = 2 \implies L = \pm\sqrt{2},$$

but we know that $x_n \geq \sqrt{2}$ hence we can ignore the negative root, and thus x_n converges to $\sqrt{2}$. ■

⊛ **Example 2.8:** ⋆

Let $a > 0$ and let (x_n) be a sequence defined recursively by x_1 is arbitrary (positive), and

$$x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}), \quad n \geq 1.$$

Show that $\lim_{n\to\infty} x_n = \sqrt{a}$.

Proof. By corollary 2.1, $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n}) \geq \sqrt{x_n \cdot \frac{a}{x_n}} = \sqrt{a}$, hence, x_n is bounded from below by $\sqrt{a}$.

We also have that $x_n - x_{n+1} = x_n - \frac{1}{2}x_n - \frac{a}{2x_n} = \frac{x_n}{2} - \frac{a}{2x_n} = \frac{1}{x_n}(x_n^2 - a)$. We have that $x_n \geq \sqrt{a} \implies x_n^2 \geq a \implies x_n^2 - a \geq 0$. Further, since the sequence is bounded from below by $\sqrt{a} > 0 (\impliedby a > 0)$, then $\frac{1}{x_n} > 0$ as well. Hence, $\frac{1}{x_n}(x_n^2 - a) \geq 0$, and thus $x_n - x_{n+1} \geq 0 \implies x_n \geq x_{n+1}$ and thus x_n is decreasing. Thus, by the Monotone Convergence Theorem, x_n is convergent. Let $X := \lim_{n\to\infty} x_n$. We have from the recursive definition, $\lim x_n = \lim \left(\frac{1}{2}(x_n + \frac{a}{x_n})\right)$. Since we know

[32] This example, as well as the more general one after it, rely on applying 1) the monotone convergence theorem, then 2) using Algebraic Limit Properties to turn the problem into an algebraic problem, using the given recursive relation.

x_n convergent, we can "split up" this limit using algebraic properties, hence

$$\lim x_n = \lim \frac{1}{2}x_n + \lim \frac{a}{2x_n} = \frac{1}{2}\lim x_n + \frac{a}{2}\lim \frac{1}{x_n}$$
$$\implies X = \frac{1}{2}X + \frac{a}{2X}$$
$$\implies \frac{X}{2} = \frac{a}{2X} \implies X^2 = a \implies X = \sqrt{a},$$

which completes the proof. ■

⊛ **Example 2.9**

Evaluate[33] the limit of x_n given the recursive relation $x_{n+1} = \frac{1}{4-x_n}$, $x_1 = 3$.

Proof. We aim to show that (x_n) is bounded from below and decreasing.

Bounded from below: we claim $x_n > 0$; we proceed by induction. $x_1 = 3 > 0$ holds; say $x_n > 0$ for some $n \geq 1$. Then, we have

$$x_n > 0 \implies -x_n < 0 \implies 4-x_n < 4 \implies \frac{1}{4-x_n} > \frac{1}{4} > 0 \implies x_{n+1} = \frac{1}{4-x_n} > 0,$$

so the sequence is bounded from below by 0.

Decreasing: (x_n) decreasing iff $x_{n+1} \leq x_n \,\forall\, n$. We have $x_2 = \frac{1}{4-3} = 1 \implies x_1 = 3 \geq 1$ holds. Say $x_{n-1} \geq x_n$ for some $n \geq 1$. Then, we have

$$x_{n-1} \geq x_n \implies 4-x_{n-1} \leq 4-x_n \implies \frac{1}{4-x_{n-1}} \geq \frac{1}{4-x_n} = x_{n+1} \implies x_n \geq x_{n+1}$$

and thus the sequence decreases, and by theorem 2.5 the limit exists. Let $X = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{4-x_{n-1}} \implies X = \frac{1}{4-X} \implies 4X - X^2 = 1 \implies 0 = X^2 - 4X + 1 \implies X = \cdots = 2 \pm \sqrt{3}$. We must take the negative root, since X is decreasing and thus must be less than 3. ■

[33] Abbott, pg 54 exercise 2.4.2

2.3 Limit Superior, Inferior

↪ **Definition 2.7: limsup, liminf**

Recall theorem 2.2, stating that a convergence sequence is bounded. Let (x_n) be a convergent sequence bounded by m and M from below/above resp, ie

$$m \leq x_n \leq M, \forall\, n$$

and let $A_n = \{x_k : k \geq n\}$ (the set of elements in the sequence "after" a particular index).

Let $y_n = \sup A_n$; by definition, $y_n \leq M$, and $y_n \geq m$, since $y_n \geq x_n \geq m$. Thus, we have

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq A_{n+1} \supseteq \cdots ,$$

and further,

$$y_1 \geq y_2 \geq \cdots \geq y_n \geq y_{n+1} \geq \cdots ;$$

since $A_2 \subseteq A_1$, y_1 also an upper bound for A_2, and thus $y_2 \leq y_1$ by definition of a supremum.

So, the sequence (y_n) is decreasing, and bounded from below; by MCT, $\lim_{n\to\infty} y_n = y$ exists. Note too that since $m \leq y_n \leq M$, we have that $m \leq y \leq M$.

This y is called the *limit superior* of (x_n) denoted by

$$\overline{\lim}_{\mathrm{n}\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Now, similarly, note that A_n is bounded below by m and thus $z_n = \inf A_n$ exists. We further have that $z_n \leq x_n \leq M$, and that $z_n \geq m \,\forall\, n$, and we have

$$z_1 \leq z_2 \leq \cdots \leq z_n \leq z_{n+1} \leq \cdots ,$$

by a similar argument as before. So, as before, the sequence (z_n) is increasing, and bounded from above by M. Again, by MCT, $\lim_{n\to\infty} z_n = z$ exists. We call z the *limit inferior* of (x_n), and denote

$$\underline{\lim}_{\mathrm{n}\to\infty} x_n = \liminf_{n\to\infty} x_n.$$

We note that $y_n \geq z_n$, so $\overline{\lim}_{\mathrm{n}\to\infty} x_n \geq \underline{\lim}_{\mathrm{n}\to\infty} x_n \quad (y \geq z)$.

Further, $\liminf$ and $\limsup$ exist for any bounded sequence, *regardless* if whether or not the limit itself exists.

⊛ **Example 2.10**

Let $(x_n) = (-1)^n, n \in \mathbb{N}$. We showed previously that this is a divergent sequence, so the limit does not exist. However, the sequence is bounded, since $-1 \leq x_n \leq 1 \,\forall\, n$. We have $A_n = \{(-1)^k : k \geq n\} = \{-1, 1\}$. So, $y_n = \sup A_n = 1$, and $z_n = \inf A_n = -1$, $\forall\, n$. Thus, $\limsup x_n = \lim y_n = 1$, and $\liminf x_n = \lim z_n = -1$, despite $\lim x_n$ not existing.

More specifically, we have a divergent sequence, and $\liminf \neq \limsup$.

↪ **Theorem 2.6:** $\liminf$**,** $\limsup$ **and convergence**

Let (x_n) be a *bounded* sequence. The following are equivalent;

1. The sequence (x_n) is convergent, and $\lim_{n\to\infty} x_n = x$.
2. $\overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n = x$.

Proof. Let A_n, y_n, z_n be as in the definition of $\limsup, \liminf$.

(1) $\implies$ (2): Suppose (x_n) is converging, and $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$. Then, there exists some $N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$|x_n - x| < \frac{\varepsilon}{2},$$

or equivalently,

$$x - \frac{\varepsilon}{2} < x_n < x + \frac{\varepsilon}{2}, \forall n \geq N.$$

Since $A_n = \{x_k : k \geq n\}$, if $n \geq N$, then $x + \frac{\varepsilon}{2}$ is an upper bound for A_n, and $x - \frac{\varepsilon}{2}$ is a lower bound for A_n. This gives that

$$y_n = \sup A_n \leq x + \frac{\varepsilon}{2}; \quad z_n = \inf A_n \geq x - \frac{\varepsilon}{2}.$$

This gives that for $n \geq N$,

$$x - \frac{\varepsilon}{2} \leq z_n \leq x_n \leq y_n \leq x + \frac{\varepsilon}{2},$$

ie $z_n, y_n \in [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$. So, for all $n \geq N$, $|z_n - x| \leq \frac{\varepsilon}{2} < \varepsilon$, and $|y_n - x| \leq \frac{\varepsilon}{2} < \varepsilon$, so by definition of the limit, this gives

$$\lim_{n\to\infty} y_n = x \text{ and } \lim_{n\to\infty} z_n = x,$$

ie, $\overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n = x$.

•

(2) $\implies$ (1): Let $\varepsilon > 0$. Since $\lim_{n\to\infty} y_n = x$, $\exists N_1$ s.t. $\forall n \geq N_1, |y_n - x| < \varepsilon$. Similarly, since $\lim z_n = x$, $\exists N_2$ s.t. $\forall n \geq N_2, |z_n - x| < \varepsilon$.
Take $N = \max\{N_1, N_2\}$. Then, for $n \geq N$, we have

$$x - \varepsilon < z_n \leq x_n \leq y_n < x + \varepsilon.$$

So, for $n \geq N$, $|x_n - x| < \varepsilon$, thus $\lim x_n = x$ as desired. ■

⊛ **Example 2.11**

Let (x_n) be a bounded sequence. Then

$$\limsup_{n\to\infty}(-x_n) = -\liminf_{n\to\infty} x_n.$$

Proof. Recall Remark 1.2; Let $A_n := \{x_k : k \geq n\}$ as in the definition of $\limsup, \liminf$. Let $y_n := \sup A_n$, $z_n := \inf A_n$. By theorem 2.6, $\lim y_n = \lim z_n$. Further, $\sup(-A_n) = -\inf(A_n)$, where $-A_n = \{-x_k : k \geq n\}$; hence, $\limsup(-x_n) = -\liminf x_n$, as desired. ■

Remark 2.3. *Given* (x_n) *bounded and* $\alpha \geq 0$, *then the following holds:*

$$\overline{\lim}_{n\to\infty}(\alpha x_n) = \alpha\,\overline{\lim}_{n\to\infty}(x_n) \quad \text{and} \quad \underline{\lim}_{n\to\infty}(\alpha x_n) = \alpha\,\underline{\lim}_{n\to\infty} x_n.$$

↪ **Proposition 2.2**

Let (x_n) and (y_n) be bounded sequences. Then,

$$\overline{\lim}_{n\to\infty}(x_n + y_n) \leq \overline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n \tag{1}$$

and

$$\underline{\lim}_{n\to\infty}(x_n + y_n) \geq \underline{\lim}_{n\to\infty} x_n + \underline{\lim}_{n\to\infty} y_n \tag{2}$$

Proof. (1) Take $A_n = \{x_k + y_k : k \geq n\}$, $B_n = \{x_k : k \geq n\}$, $C_n = \{y_k : k \geq n\}$. Then, take

$$B_n + C_n = \{x_k + y_j : k \geq n, j \geq n\} \supseteq A_n$$

and so $\sup A_n \leq \sup(B_n + C_n)$. We have shown previously (assignment question) that $\sup(B_n + C_n) = \sup B_n + \sup C_n$. Let now

$$t_n = \sup A_n \quad r_n = \sup B_n \quad s_n = \sup C_n,$$

so $t_n \leq r_n + s_n$, that is, $\lim t_n \leq \lim r_n + \lim s_n$, and thus $\overline{\lim}_{n\to\infty}(x_n + y_n) \leq \overline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n$, proving (1).

(2) The same argument holds, replacing each instance of $\overline{\lim}_{n\to\infty}$ with $\underline{\lim}_{n\to\infty}$ and reversing inequalities where necessary. Alternatively, it follows directly from (1) by negating the sequences where appropriate. ■

↪ **Proposition 2.3**

Let (x_n) be a bounded sequence. Then

1. $\overline{\lim}_{n\to\infty} x_n = \inf\{t : \{n : x_n > t\}$ is either empty or finite $\}$
2. $\underline{\lim}_{n\to\infty} x_n = \sup\{t : \{n : x_n < t\}$ is either empty or finite $\}$

Remark 2.4. *(2) follows from (1) by either repeating the argument used to prove (1) (changing notation), or using the fact that* $\underline{\lim}_{n\to\infty} x_n = -\overline{\lim}_{n\to\infty}(-x_n)$.

Remark 2.5. *The set* $\{n : x_n > t\}$ *is empty or finite iff* $\exists\, n_t \in \mathbb{N}$ *s.t.* $\forall\, n > n_t, x_n \leq t$. *The set is empty or finite if* t *is an* eventual upper bound *for* (x_n)*; that is, starting with sufficiently large* n_t, $x_n \leq t \,\forall\, n \geq t$.

In other words, t *is an upper bound if we neglect finitely many elements. Hence, (1) states equivalently states that* $\overline{\lim}_{n\to\infty} x_n$ *is the infimum of the eventual upper bounds for* (x_n)*;*

$$\{n : x_n > t\} \text{ empty or finite } \iff$$
$$\begin{cases} \text{empty} & x_n \leq t \,\forall\, n \iff t \text{ an upper bound of } x_n \\ \text{finite} & t \text{ upper bounds } x_n \text{ for an infinite number of } n\text{'s} \end{cases}$$

Proof. (Of (1)) Let $A = \{t : \{n : x_n > t\}$ is either empty or finite $\}$. We note that this set is non-empty and bounded from below, hence the inf is well-defined. We can see this by recalling that (x_n) bounded, hence $\forall\, n\, \exists m, M$ s.t. $m \leq x_n \leq M$. Then, $\{n : x_n > M\}$ is empty, hence $M \in A$. Otoh, if $t < m$, then the set $\{n : x_n > t\} = \mathbb{N}$ since $x_n \geq m > t \,\forall\, n$. So, if $t < m$, then $t \notin A$ and hence m is a lower bound for A.

We have now that $\overline{\lim}_{n\to\infty} x_n$ is a lower bound for A, and hence $\overline{\lim}_{n\to\infty} x_n \geq \inf A$. Let $t \in A$. We aim to show that $\overline{\lim}_{n\to\infty} x_n \leq t$.

The set $\{n : x_n > t\}$ is finite by definition; assume $t \in A$. We can then let

$$n_t = \max\{n : x_n > t\}.$$

Then, if $k > n_t$, it must be that $x_k \leq t$. Consider now $n > n_t$, then $y_n = \sup\{x_k : k \geq n\}$ and since $x_k \leq t$ for $k \geq n$, and t upper bounds $\{x_k : k \geq n\}$, we have that $y_n \leq t$ for $n > n_t$. Hence, for sufficiently large n, $y_n \leq t$, thus $\lim y_n \leq t \implies \overline{\lim}_{n\to\infty} x_n \leq t$.

Thus, $\overline{\lim}_{n\to\infty} x_n \leq \inf A$. ■

Proof. (Of (1), More Concise) Let $\alpha = \inf A$ and $\varepsilon > 0$. Then $\alpha - \varepsilon$ is not in A, $\alpha - \varepsilon$ is not an eventual upper bound for x_n. Since $\alpha - \varepsilon \notin A$, the set $\{n : x_n > \alpha - \varepsilon\}$ is infinite.[34]. Hence, for any m we can find n such that $n \geq m$ and $x_n > \alpha - \varepsilon$.

[34] This is important; understand why this is

Let, now,

$$y_m = \sup\{x_n : n \geq m\}.$$

By our last observation, we have that $y_m > \alpha - \varepsilon$. By Order Properties of Limits,

$$\overline{\lim}_{n\to\infty} x_n = \lim_{m\to\infty} y_n \geq \alpha - \varepsilon,$$

so for any $\varepsilon > 0$, $\overline{\lim}_{n\to\infty} x_n \geq \alpha - \varepsilon$, and thus $\overline{\lim}_{n\to\infty} x_n \geq \alpha = \inf A$, and the proof is complete. ■

2.4 Subsequences and Bolzano-Weirestrass Theorem

↪ **Definition 2.8: Subsequence**

Let (x_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$ be a strictly increasing sequence of natural numbers. Then, the sequence

$$(x_{n_1}, x_{n_2}, \cdots, x_{n_k}, x_{n_{k+1}}, \cdots)$$

is called a *subsequence* of (x_n) and is denoted $(x_{n_k})_{k\in\mathbb{N}}$.

Remark 2.6. *k is the index of the subsequence, $(x_{n_k})_{k\in\mathbb{N}}$, **not** n; x_{n_1} is the 1st element, …, x_{n_k} is the k-th element.*

⊛ **Example 2.12**

Let $x_n = \frac{1}{n}, (\frac{1}{n})_{n\in\mathbb{N}}$, and let $n_k = 2k+1, k \in \mathbb{N}$. $n_1 = 3, n_2 = 5, n_3 = 7, \ldots, n_k = 2k+1$. Our subsequence is then

$$(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots) = \left(\frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2k+1}, \cdots\right) = \left(\frac{1}{2k+1}\right)_{k\in\mathbb{N}}$$

is our subsequence of (x_n).

Remark 2.7. *Note that for any $k, n_k \geq k$.*
Let $S = \{k \in \mathbb{N} : n_k \geq k\}$. Then, $1 \in S$, since $n_1 \in \mathbb{N}, n_1 \geq 1$. If $k \in S$, then $n_k \geq k$, and so, since $n_{k+1} > n_k$ (increasing), we have that $n_{k+1} > k \implies n_{k+1} \geq k+1$. So, $k+1 \in S, S = \mathbb{N}$.

Remark 2.8. $\lim_{k\to\infty} x_{n_k} = x$ *if* $\forall\, \varepsilon > 0, \exists K \in \mathbb{N}$ *s.t.* $\forall\, k \geq K, |x_{n_k} - x| < \varepsilon$.

↪ **Theorem 2.7**

Let (x_n) be a sequence such that $\lim_{n\to\infty} x_n = x$. Then, for any subsequence $(x_{n_k})_{k\in\mathbb{N}}$, we have that $\lim_{k\to\infty} x_{n_k} = x$

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|x_n - x| < \varepsilon$. Take $K = N$ (from Remark 2.8). Then, for $k \geq K$, we have from Remark 2.7 that

$$n_k \geq k \geq K = N,$$

and hence $|x_{n_k} - x| < \varepsilon \implies \lim_{k\to\infty} x_{n_k} = x$. ■

↪ **Theorem 2.8: Bolzano-Weirestrass Theorem**

[35] Any bounded sequence (x_n) has a convergent subsequence.

[35] Fundamental property of the real line; equivalent to AC.

⊛ **Example 2.13**

Take $x_n = (-1)^n, n \in \mathbb{N}$. This sequence does not converge. However, if we take a subsequence with $n_k = 2k, k \in \mathbb{N}$. $x_{n_k} = (-1)^{2k} = 1$, so (x_{n_k}) is a constant sequence 1 and converges to 1.

Similarly, if $n_k = 2k+1, k \in \mathbb{N}$, then $x_{n_k} = (-1)^{2k+1} = -1$, and the subsequence converges to -1.

↪ **Proposition 2.4**

If $0 < b < 1$, then $\lim_{n\to\infty} b^n = 0$.

Proof. Let $x_n = b^n$. Then $x_n > 0$, and $x_{n+1} = b^{n+1} = bx_n < x_n$, and since $0 < b < 1$, (x_n) is decreasing and bounded from below, (x_n) converges by the Monotone Convergence Theorem. Let $x = \lim_{n\to\infty} x_n$. Again, $x_{n+1} = bx_n$, so $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} bx_n = b\lim_{n\to\infty} x_n$, so $x = bx \implies (1-b)x = 0$. $0 < b < 1 \implies x = 0$. ■

BW Proof (1): using Nested Interval Property. [36] Since (x_n) bounded, $\exists M > 0$ s.t. $|x_n| \leq M \,\forall n \in \mathbb{N}$. Let $I_1 = [-M, M]$ and $n_1 = 1$. We now construct I_2, n_2 as follows.

Divide I_1 into two intervals of the same size, $I_1' = [-M, 0], I_1'' = [0, M]$. Now, consider the sets

$$A_1 = \{n \in \mathbb{N} : n > n_1(= 1), x_n \in I_1'\}, \quad A_2 = \{n \in \mathbb{N} : n > n_1, x_n \in I_1''\}$$

(ie, all the indices of all the elements in I_1', I_1'' resp.).

Hence, $A_1 \cup A_2 = \{n : n > n_1\}$, an infinite set, and hence, one of A_1, A_2 must be infinite (by theorem 1.9). If A_1 infinite, set $I_2 = I_1'$, $n_2 = \min A_1$. If A_1 finite, then A_2 infinite, and set $I_2 = I_1''$, $n_2 = \min A_2$.

Suppose now that I_k, n_k are chosen, and that I_k contains infinitely many elements of the sequence (x_n). Divide I_k into two equal sub-intervals, I_k', I_k''. We now introduce

$$A_1^{(k)} = \{n \in \mathbb{N} : n > n_k \text{ and } x_n \in I_k'\}, \quad A_2^{(k)} = \{n \in \mathbb{N} : n > n_k \text{ and } x_n \in I_k''\},$$

(similar to our construction of A_1, A_2). $A_1^{(k)} \cup A_2^{(k)}$ must be infinite, so one of the two must be infinite. If A_1 infinite, set $I_{k+1} = I_k'$, $n_{k+1} = \min A_1^{(k)}$. If A_2 infinite, set $I_{k+1} = I_k''$, $n_{k+1} = \min A_2^{(k)}$.

This gives now that I_{k+1} and n_{k+1}, where $I_{k+1} \subseteq I_k$, I_{k+1} contains infinitely many elements of the sequence. Further, by construction, $n_{k+1} > n_k$. This gives us a sequence of closed intervals $I_k = [a_k, b_k]$, $k \in \mathbb{N}$ such that $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq I_{k+1} \supseteq \cdots$, such that $x_{n_k} \in I_k$, and that n_k is a strictly increasing sequence of natural numbers, defining subsequence (x_{n_k}).

Now, by construction, the length of I_{k+1} is $\frac{1}{2}$ of the length of I_k. Since $I_k = [a_k, b_k]$, then

$$b_k - a_k = \frac{b_{k-1} - a_{k-1}}{2} = \cdots \frac{b_1 - a_1}{2^{k-1}} = \frac{2M}{2^{k-1}} = \frac{M}{2k^{k-2}}.$$

Since I_k, $k \in \mathbb{N}$, is a nested sequence of closed intervals and by the nested interval property of the real line (AC), $\exists x \in \bigcap_{k=1}^{\infty} I_k$.

We claim now that our subsequence (x_{n_k}) satisfies $\lim_{k\to\infty} x_{n_k} = x$. To see this, let $\varepsilon > 0$. Since $\lim_{k\to\infty} \frac{M}{2^{k-2}} = \lim_{k\to\infty} \frac{4M}{2k} = 0$, by proposition 2.4, with $b = \frac{1}{2}$. There exists $K \in \mathbb{N}$ such that $\forall\, k \geq K$, we have $\frac{M}{2^{k-2}} = b_k - a_k < \varepsilon$. So, since I_k is a nested sequence of intervals, $\forall\, k \geq K$, $x_{n_k} \in I_K$ ($x_{n_k} \in I_k \subseteq I_K$). We also have that $x \in I_K$, since $x \in \bigcap I_k$. So, $x, x_{n_k} \in [a_K, b_K]\, \forall\, k \geq K$. So, for $k \geq K$, $|x_{n_k} - x| \leq |b_k - a_k| < \varepsilon$. So for $\varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\forall\, k \geq K$, $|x_{n_k} - x| < \varepsilon$, and so $\lim_{k\to\infty} x_{n_k} = x$, as desired. ∎

[36]Sketch: this proof is somewhat diagonal in nature (if one can say that); if you understand the proof of Cantor's Theorem using the Nested Interval property, this should follow naturally. In short, construct subsequences such that the subsequence has all its terms contained in a "nest" of intervals, and show that the length (sts) of these intervals converges to 0. But these are subsets of $\mathbb{R}$, their intersect must contain some element, show that this

↪ **Definition 2.9: Peak**

Let (x_n) be a sequence of real numbers. An element x_m is called a *peak* of this sequence if $x_m \geq x_n \,\forall\, n \geq m$. x_m is bigger or equal then to any element of the sequence that follows it.

If a sequence is decreasing, then any element of the sequence is a peak.

If a sequence is increasing, then there is no peak.

BW Proof (2): using Peaks. Take sequence (x_n). Then,

- **Case 1:** (x_n) has *infinitely* many peaks; enumerate the indices of those peaks as $n_1 < n_2 < n_3 < \cdots$, then $x_{n_k} < x_{n_{k+1}} \forall k$, since x_{n_k} is a peak, $n_{k+1} > n_k$. This gives a decreasing subsequence (x_{n_k}).
- **Case 2:** (x_n) has *finitely* many peaks, with indices $m_1 < m_2 < \cdots < m_r$. Set $n_1 = m_r + 1$. Then x_{n_1} is not a peak, and so $\exists\, n_2 > n_1$ s.t. $x_{n_2} > x_{n_1}$. Now, x_{n_2} is also not a peak, ($n_2 > n_1 > m_r$), and so there exists $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$, and so on. In this way, we construct a subsequence (x_{n_k}) that is strictly increasing, that is, $x_{n_{k+1}} > x_{n_k}$.

If in addition (x_n) is bounded, say $|x_n| \leq M \forall n$, then the monotone subsequence constructed in **Cases 1, 2** is also bounded; ie $|x_{n_k}| \leq M \forall k$. Thus, by Monotone Convergence Theorem, (x_{n_k}) is converging. ■

2.5 Cauchy Sequences

↪ **Definition 2.10: Cauchy Sequence**

A sequence (x_n) is called *Cauchy* if for every $\varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N, |x_n - x_m| < \varepsilon$.

↪ **Theorem 2.9: Cauchy Criterion**

A sequence (x_n) is convergent iff it is Cauchy.[37]

Remark 2.9. *This is, again, an "equivalent" formulation of AC; at least, the direction* (x_n) *Cauchy* $\implies$ *convergent is. The other direction, convergent* $\implies$ *Cauchy, does not rely on AC.*

Remark 2.10. *AC* $\iff$ *BW, AC* $\iff$ *MCT, AC* $\iff$ *NIP; AC* $\iff$ *Cauchy Criterion* $+$ *Archimedean Property*

Remark 2.11. *Beyond the real line, AC (in terms of* $\sup$*) cannot be formulated, because of the lack of ordering. In this case, the Cauchy criterion can be used to extend AC to other spaces.*

Proof. (theorem 2.9; (x_n) **Convergent** $\implies$ **Cauchy**)
Suppose $\lim_{n\to\infty} x_n = x$. Let $\varepsilon > 0$, $N \in \mathbb{N}$ s.t. $\forall n \geq N$, $|x_n - x| < \frac{\varepsilon}{2}$. Then, for

[37] Sketch: Convergent $\implies$ Cauchy; use definition of Cauchy, add/subtract limit of sequence, triangle inequality (and choose your ε to be $\varepsilon/2$, optional).
Convergent $\impliedby$ Cauchy; show that any Cauchy sequence is bounded (theorem 2.11), and thus has a converging subsequence (Bolzano-Weirestrass Theorem); finally, show that any Cauchy sequence that has a converging subsequence itself converges (theorem 2.10), and you are done.

$n, m \geq N$,

$$|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x_m - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
$$\implies |x_n - x_m| < \varepsilon,$$

hence (x_n) is Cauchy. ■

Remark 2.12. *To prove* $\Longleftarrow$ *, we first introduce the following theorem(s); see section 2.5 for the remainder.*

↪ **Theorem 2.10**

Let (x_n) be a Cauchy sequence and suppose that (x_n) has a convergent subsequence (x_{n_k}). Then (x_n) is also convergent.

Proof. Let $x = \lim_{n\to\infty} x_{n_k}$. Let $\varepsilon > 0$. Then, $\exists K \in \mathbb{N}$ such that $\forall k \geq K$, $|x_{n_k} - x| < \varepsilon$. We have too that (x_n) Cauchy, ie $\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$, $|x_n - x_m| < \frac{\varepsilon}{2}$.
Let now $K_0 \geq \max\{K, N\}$. Recall that $n_{K_0} \geq K_0 \geq N$. Take now $n \geq N$, and estimate

$$|x_n - x| = \left|x_n - x_{n_{K_0}} + x_{n_{K_0}} - x\right| \leq \left|x_n - x_{n_{K_0}}\right| + \left|x_{n_{K_0}} - x\right|$$

Since $K_0 \geq K$, $\left|x_{n_{K_0}} - x\right| < \frac{\varepsilon}{2}$. Since $n_{K_0} \geq N$, we also have $\left|x_n - x_{n_{K_0}}\right| < \frac{\varepsilon}{2}$. Thus, we have

$$|x_n - x| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

hence $\lim_{n\to\infty} x_n = x$. ■

Remark 2.13. *This did not use AC.*

↪ **Theorem 2.11**

Any Cauchy sequence is bounded.

Proof. Let (x_n) be Cauchy. We aim to show that $\exists M > 0$ s.t. $\forall n \in \mathbb{N}$, $|x_n| \leq M$.
Take $\varepsilon = 1$ in the definition of Cauchy sequence. Let N be such that $\forall n, m \geq N$, $|x_n - x_m| < 1$. We can take $m = N$, and so for all $n \geq N$, $|x_n - x_N| < 1$, which gives that for $n \geq N$,

$$|x_n| = |x_n - x_N + x_N| \leq |x_n - x_N| + |x_N| < 1 + |x_N|$$

Let[38]

[38] While this seems like an arbitrary definition, this is a common "trick" to find a bound of a sequence based on

$$M = |x_1| + |x_2| + \dots |x_{N-1}| + |x_N| + 1.$$

Then, if $n \leq N$, $|x_n| \leq M$; if $n \geq M$, $|x_n| \leq M$, so $\forall n \geq 1, |x_n| \leq M$, hence (x_n) is bounded. ■

Remark 2.14. *This did not use AC.*

Proof. (theorem 2.9; (x_n) **Convergent** $\Longleftarrow$ **Cauchy**)

If (x_n) Cauchy, then (x_n) is bounded by theorem 2.11, and thus by Bolzano-Weirestrass Theorem, (x_n) has a convergent subsequence (x_{n_k}). Then, by theorem 2.10, (x_n) must converge. ■

⊛ **Example 2.14**

Let[39] (x_n) be a sequence defined recursively by $x_1 = 1$, $x_2 = 2$, $x_{n+1} = \frac{1}{2}(x_n + x_{n-1}), n \geq 2$. Prove that (x_n) is a convergence sequence, and find its limit.

[39] Sketch: show x_n Cauchy $\Longrightarrow$ x_n converges, then take a subsequence of x_n (spec, odd n) and find a closed form of it which is nicer to evaluate. Use then theorem 2.7 to conclude that the limit of the subsequence is equal to the limit of the sequence.

Remark 2.15. *Before solving, we establish a number of properties about the sequence.*

↪ **Proposition 2.5: Property I**

$1 \leq x_n \leq 2 \, \forall \, n \geq 1$

Proof. We proceed by induction. Let $S \subseteq \mathbb{N}$ be the set of all n such that $1 \leq x_n \leq 2$.

Base Case: $1 \in x$, since $x_1 = 1$.

Assumption: suppose $\{1, 2, \dots, n\} \in S$. We want to show that $n + 1 \in S$.

If $n = 1$, then $x_2 = 2$, so $x_2 \in S$. If $n > 1$, then

$$x_{n+1} = \frac{1}{2}(x_n + x_{n+1}),$$

and by inductive assumption, $1 \leq x_n \leq 2$ and $1 \leq x_{n-1} \leq 2$, hence

$$1 \leq x_{n+1} \leq 2,$$

hence $n + 1 \in S$, and thus $S = \mathbb{N}$. ■

↪ **Proposition 2.6: Property II**

For all $n \geq 1$, $|x_{n+1} - x_n| = \frac{1}{2^{n-1}}$.

Proof. We proceed by induction. Let $S \subseteq \mathbb{N}$ be the set of all n such that the statement holds for x_n.

Base Case: $x_2 = 2$, $x_1 = 1$, hence $2 - 1 = 1 = \frac{1}{2^0} = 1$, holds.

Assumption: suppose $n \in S$, ie $|x_{n+1} - x_n| = \frac{1}{2^{n-1}}$ holds for n. Then,

$$\begin{aligned} |x_{n+2} - x_{n+1}| &= \left|\frac{1}{2}(x_{n+1} + x_n) - x_{n+1}\right| \\ &= \left|\frac{1}{2}x_n - \frac{1}{2}x_{n+1}\right| = \frac{1}{2}|x_{n+1} - x_n| \\ \text{(assumption} \implies \text{)} \qquad &= \frac{1}{2} \cdot \frac{1}{2^{n-1}} = \frac{1}{2^n}, \end{aligned}$$

hence the statement holds for $n + 1$, and $S = \mathbb{N}$. ■

↪ **Corollary 2.2**

For any $r \neq 1$, and any $k \in \mathbb{N}$, $1 + r + r^2 + \cdots + r^k = \frac{1-r^{k+1}}{1-r}$.

Proof. We proceed by induction. $k = 1 \implies r^0 = \frac{1-r^1}{1-r} = 1$, holds. Suppose $1 + \cdots r^{k-1} = \frac{1-r^k}{1-r}$ holds for some $k - 1 \in \mathbb{N}$. Then, we have that

$$\begin{aligned} 1 + \cdots r^{k-1} + r^k = \frac{1-r^k}{1-r} + r^k &= \frac{1 - r^k + (1-r)r^k}{1-r} \\ &= \frac{1 \cancel{- r^k} + \cancel{r^k} - r^{k+1}}{1-r} \\ &= \frac{1 + 1 - r^{k+1}}{1-r}, \end{aligned}$$

hence, the statement for $k - 1 \implies$ the statement for k, hence it holds $\forall\, k \in \mathbb{N}$ and the proof is complete. ■

↪ **Proposition 2.7: Property III**

(x_n) a Cauchy sequence.

Proof. Let $\varepsilon > 0$. We need to find $N \in \mathbb{N}$ such that $\forall\, n, m \geq N$, $|x_n - x_m| < \varepsilon$. Let N be such that[40] $\frac{1}{2^{N-2}} = \frac{4}{2^N} < \varepsilon$. Let, now, $n, m \geq N$, and suppose $n > m$ (when $n = m$, we are done; when $n < m$, simply switch the variables wlog). We can write

$$\begin{aligned} |x_n - x_m| = |x_n - x_{n-1} + x_{n+1} - x_{n-2} + x_{n-2} + \cdots - x_{m+1} + x_{m+1} - x_m| \\ \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|^{41} \end{aligned}$$

Using Property II we can write

$$
\begin{aligned}
|x_n - x_m| &\le \frac{1}{2^{m-1}} + \frac{1}{2^m} + \cdots \frac{1}{2^{n-2}} \\
&= \frac{1}{2^{m-1}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-1}}\right)
\end{aligned}
$$

By corollary 2.2, with $r = \frac{1}{2}$ and $k = n - m - 1$, we have

$$
\frac{1}{2^{m-1}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-m-1}}\right) = \frac{1}{2^{m-1}}\left(\frac{1 - \left(\frac{1}{2}\right)^{n-m}}{1 - \frac{1}{2}}\right) < \frac{1}{2^{m-2}} \le \frac{1}{2^{N-2}}.
$$

We have chosen N so that $\frac{1}{2^{N-2}} < \varepsilon$, hence for $n, m \ge N$, $|x_n - x_m| < \varepsilon$, and thus our sequence is Cauchy, so $\lim_{n\to\infty} x_n = X$ exists. ■

[36] $\lim \frac{1}{2^n} = 0$, so such an N exists.

[37] "Telescoping" the sequence; the inequality follows directly from the triangle inequality.

Proof. (Of example 2.14)
By Property III, the limit $\lim x_n = X$ exists. From the recursive definition, we can write

$$
\begin{aligned}
X = \lim x_n &= \lim\left(\frac{1}{2}(x_{n-1} + x_{n-2})\right) \\
\implies X &= \frac{1}{2}(X + X) = X,
\end{aligned}
$$

which, while true, is useless. Rather, consider the subsequence

$$
(x_{2k+1})_{k\in\mathbb{N}}
$$

of (x_n). We claim, then, that

$$
x_{2k+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{k-1}}, k \ge 1. \qquad \star
$$

Note that $\forall\, n \ge 1$, $x_{2n} \ge x_{2n-1}$ and $x_{2n} \ge x_{2n+1}$. We can argue by induction. Let $S \subseteq \mathbb{N}$ for which the relation holds. Since $x_1 = 1$, $x_2 = 2$, $x_3 = \frac{3}{2}$, we have that $x_2 \ge x_1$, $x_2 \ge x_3$, and so the relation holds, ie $1 \in S$. Suppose that $n \in S$, ie $x_{2n} \ge x_{2n-1}, x_{2n} \ge x_{2n+1}$ for some $n \ge 1$. We can write

$$
\begin{aligned}
x_{2k+2} &= \frac{1}{2}(x_{2k+1} + x_{2k}) \ge \frac{1}{2}(x_{2n+1} + x_{2n+1}) \ge x_{2n+1} \\
\implies x_{2n+3} &= \frac{1}{2}(x_{2n+2} + x_{2n+1}) \le \frac{1}{2}(x_{2n+2} + x_{2n+2}) = x_{2n+2}
\end{aligned}
$$

Hence $x_{2n+2} \ge x_{2n+1}$ and $x_{2n+2} \ge x_{2n+3}$, $n + 1 \in S$, and hence $S = \mathbb{N}$, and our relation holds $\forall\, n \in \mathbb{N}$.

Recall now that $\forall\, n, |x_{n+1} - x_n| = \frac{1}{2^{n-1}}$. We then have the following, given the relation we proved above;

$$x_{2n+1} - x_{2n-1} = \underbrace{x_{2n+1} - x_{2n}}_{\leq 0} + \underbrace{x_{2n} - x_{2n-1}}_{\geq 0}$$
$$= -\frac{1}{2^{2n-1}} + \frac{1}{2^{2n-2}} = -\frac{1}{2^{2n-1}} + \frac{2}{2^{2n-1}} = \frac{1}{2^{2n-1}}$$

From here, we can prove the claim $\star$ by induction.

Summing up the RHS of $\star$, and factoring out a $\frac{1}{2}$, we have

$$x_{2k+1} = 1 + \frac{1}{2}\left(1 + \frac{1}{2^2} + \cdots + \left(\frac{1}{2^2}\right)^{k-1}\right).$$

Recalling corollary 2.2, and taking $r = \frac{1}{4}$ and $\ell = k - 1$, we have

$$\begin{aligned} x_{2k+1} &= 1 + \frac{1}{2}\left(\frac{1 - \left(\frac{1}{4}\right)^k}{1 - \frac{1}{4}}\right) \\ &= 1 + \frac{2}{3}\left(1 - \left(\frac{1}{4}\right)^k\right) \\ &= \frac{5}{3} - \frac{2}{3}\left(\frac{1}{4}\right)^k \end{aligned}$$

Thus, we have that $\lim_{k\to\infty} x_{2k+1} = \frac{5}{3}$, as the term $(\frac{1}{4})^k$ goes to zero.

Now, since $\lim_{n\to\infty} x_n = X$ and we showed x_n convergent, then each of its subsequences converges to the same limit. Thus, $X = \frac{5}{3}$, ie,

$$\lim_{n\to\infty} x_n = \frac{5}{3}.$$

■

Remark 2.16. *Generally, this type of approach is quite tedious. The next example(s) will try to generalize it.*

⊛ **Example 2.15**

Consider the recursive relation $x_{n+1} = \frac{1}{2}x_n + \frac{1}{2}x_{n-1} \quad \star$.

Proof. We have the following *characteristic equation* of the sequence:

$$x^2 = \frac{1}{2}x + \frac{1}{2},$$

with solutions $a = 1, b = -\frac{1}{2}$. We can now write the following sequence:

$$x_n = C_1 a^n + C_2 b^n = C_1 + C_2(-\frac{1}{2})^n, \quad \star\star$$

where C_1, C_2 are arbitrary constants. We claim that this sequences satisfies our recursive relation, $\star$; note that

$$\begin{aligned}
\left(-\frac{1}{2}\right)^{n+1} &= \left(-\frac{1}{2}\right)^{n-1} \cdot \frac{1}{4} = \left(-\frac{1}{2}\right)\left(\left(-\frac{1}{2}\right)\frac{1}{2} + \frac{1}{2}\right) \\
\implies x_{n+1} &= C_1 + C_2\left(-\frac{1}{2}\right)^{n+1} \\
&= \frac{C_1}{2} + \frac{C_1}{2} + C_2\left(-\frac{1}{2}\right)^{n-1}\left(\left(-\frac{1}{2}\right)\frac{1}{2} + \frac{1}{2}\right) \\
&= \frac{C_1}{2} + \frac{C_1}{2} + C_2\left(-\frac{1}{2}\right)^{n} + \frac{C_2}{2}\left(-\frac{1}{2}\right)^{n} \\
&= \frac{C_1}{2} + \frac{C_2}{2}\left(-\frac{1}{2}\right)^{n} + \frac{C_1}{2} + \frac{C_2}{2}\left(-\frac{1}{2}\right)^{n-1} \\
&= \frac{x_n}{2} + \frac{x_{n-1}}{2}
\end{aligned}$$

Hence, our $\star\star$ is our so-called *general solution* to $\star$. The only factor we must find, then, are our C_1, C_2. Recall our initial $x_1 = 1, x_2 = 2$. Plugging these into $\star\star$, then, gives

$$x_1 = C_1 + C_2\left(-\frac{1}{2}\right) = 1; \quad x_2 = C_1 + C_2\left(-\frac{1}{2}\right)^2 = 2,$$

which is simply a system of two equations for two unknowns, C_1, C_2. Solving them[42], we have

$$C_1 = \frac{5}{3}, \quad C_2 = \frac{4}{3},$$

hence we have the general formula

$$x_n = \frac{5}{3} + \frac{4}{3}\left(-\frac{1}{2}\right)^n$$

The RHS of this sum goes to zero, and thus our limit is

$$\lim_{n\to\infty} x_n = \frac{5}{3}.$$

■

Remark 2.17. *From this general form, we can conclude, as in example 2.14, that* $x_{2n} \geq x_{2n-1}, x_{2n} \geq x_{2n+1}$, *since* $x_{2n} > \frac{5}{3}, x_{2n+1} < \frac{5}{3}, x_{2n-1} < \frac{5}{3}$; *ie, the same property that we used to prove the previous example holds here.*

Remark 2.18. *Any recursively defined sequence of the form* $x_{n+1} = Ax_n + Bx_{n-1}, n > 1$

where $A, B \in \mathbb{R}$, can be solved using the characteristic equation

$$x^2 = Ax + B,$$

with solutions $a = \frac{A+\sqrt{A^2+4B}}{2}, b = \frac{A-\sqrt{A^2+4B}}{2}$. It may be that $a, b \in \mathbb{C}$; we shall not consider these cases. Indeed, we have:

$$\begin{aligned} x_{n+1} &= C_1 a^{n+1} + C_2 b^{n+1} \\ &= \cdots \\ &= C_1 a^{n-1}(Aa + B) + C_2 b^{n-1}(Ab + B) \\ &= C_1 A a^n + C_1 a^{n-1} B + C_2 A b^n + C_2 B b^{n-1} \\ &= A(C_1 a^n + C_2 b^n) + B(C_1 a^{n-1} + C_2 b^{n-1}) \\ &= A x_n + B x_{n-1} \end{aligned}$$

Given initial x_1, x_2, then we have that

$$x_1 = C_1 a + C_2 b, \quad x_2 = C_2 a^2 + C_2 b^2.$$

C_1, C_2 are uniquely determined by this relation, as long as the matrix of coefficients

$$\begin{vmatrix} a & b \\ a^2 & b^2 \end{vmatrix} = ab^2 - ba^2 \neq 0.$$

In the case $a = b$, or $a = 0$ or $b = 0$, then the determinant is also equal to 0, and we thus have to use a different method. As long as the determinant is nonzero, then we have a valid specific definition.

Remark 2.19. *Note that nothing in this derivation assumed x_n convergent; this form can indeed be found even if x_n diverges. It will simply also diverge.*

Remark 2.20. *The recursive relation $x_{n+1} = Ax_n + Bx_{n-1}$ is a discrete analog of a differential equation.*

2.6 Contractive Sequences

↪ **Definition 2.11: Contractive Sequences**

A sequence (x_n) of real numbers is called contractive with contractive constant K, where $0 < K < 1$, if $|x_{n+2} - x_{n+1}| \leq K|x_{n+1} - x_n| \, \forall n \geq 1$, ie, the distance between successive elements of the sequence are contracted at least by a factor of K.

We have, by extension, that

$$
\begin{aligned}
|x_n - x_{n-1}| &\leq K|x_{n-1} - x_{n-2}| \\
&\leq K^2|x_{n-2} - x_{n-3}| \\
&\leq \cdots \\
&\leq K^{n-2}|x_2 - x_1|.
\end{aligned}
$$

↪ **Theorem 2.12**

Let[43] (x_n) be a contractive sequence with contractive constant K. Then, (x_n) is a Cauchy sequence, and in particular, (x_n) converges.

[43] Sketch: start with $|x_n - x_m|$, and add/subtract each term between x_n and x_m, use triangle inequality, "substitute" in contractive constant, collect like terms, and simplify. This creates an upper bound for $|x_n - x_m|$, which converges to 0, then use this converges to define the epsilon to use in the Cauchy definition.

Proof. Let $n, m \in \mathbb{N}$ such that $n > m \geq 2$. Then, we have

$$
\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots - x_{m+1} + x_{m+1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&\leq K^{n-2}|x_2 - x_1| + K^{n-3}|x_2 - x_1| + \cdots + K^{m-1}|x_2 - x_1| \\
&= K^{m-1}|x_2 - x_1| \left(1 + K + K^2 + \cdots + K^{n-m-1}\right) \\
&= K^{m-1}|x_2 - x_1| \frac{1 - K^{n-m}}{1 - K} \quad \textit{by corollary 2.2} \\
&< \frac{K^{m-1}|x_2 - x_1|}{1 - K} \\
&\implies |x_n - x_m| < \frac{K^{m-1}|x_2 - x_1|}{1 - K} \, \forall \, n > m \geq 2
\end{aligned}
$$

$$
\begin{aligned}
&\lim \frac{K^{m-1}}{1 - K}|x_2 - x_1| = 0 \implies \forall \, \varepsilon > 0, \exists N \text{ s.t. } \forall \, m > N, \frac{K^{m-1}}{1 - K}|x_2 - x_1| < \varepsilon \\
&\to n > m \geq N \implies |x_n - x_m| \leq \frac{K^{m-1}}{1 - K}|x_2 - x_1| < \varepsilon \\
&\to m > n \geq N \implies |x_n - x_m| \leq \frac{K^{n-1}}{1 - K}|x_2 - x_1| < \varepsilon \\
&\implies \forall \, m, n \geq N, |x_m - x_n| < \varepsilon, \text{ and } (x_n) \text{ Cauchy}
\end{aligned}
$$

■

Remark 2.21. *This proof also gives us a* rate of convergence*; we have*

$$
|x_n - x_m| \leq \frac{K^{m-1}}{1 - K} \cdot |x_2 - x_1|,
$$

together with the fact that $\lim_{n \to \infty} x_n = X$*, whose convergence also implies by* Algebraic

Properties of Limits that

$$\lim |x_n - x_m| = |X - x_m|.$$

This implies, by Order Properties of Limits, that

$$|X - x_m| \leq \frac{K^{m-1}}{1-K}|x_2 - x_1|,$$

that is, the sequence converges exponentially fast.

Remark 2.22. *We have that* $\lim_{n\to\infty} |x_n - x_m| = |X - x_m|$ *where* $(x_n) \to X$*, by the inequality*

$$||X - x_m| - |x_n - x_m|| \leq |x - x_n| < \varepsilon,$$

following from the more general fact that

$$||a| - |b|| \leq |a - b| \quad \forall\, a, b \in \mathbb{R},$$

a direct consequence of the Triangle Inequality detailed in lemma 2.1.

Remark 2.23. *The result that every contractive sequence is convergent is a simple example of the more general* "Fixed Point Theorems"*; this proof can be generalized to the* Banach Fixed Point Theorem *on arbitrary metric spaces. This is further used to establish the existence and uniqueness of solutions of differential, integral equations.*[44]

[44] See the Picard-Lindelöf Theorem

Remark 2.24. *In the case of the recursively defined*

$$x_{n+1} = \frac{1}{2}(x_n + x_{n-1}),$$

we have that

$$|x_{n+2} - x_{n+1}| > \frac{1}{2}|x_{n+1} - x_n|,$$

that is, x_n *is a contractive sequence with* $K = \frac{1}{2}$*. The argument used to prove that this inequality implies* (x_n) *Cauchy is the same as the one we used to prove a general contractive sequence is Cauchy.*

⊛ **Example 2.16**

Let (x_n) be a sequence defined recursively by $x_1 = 2$, $x_{n+1} = 2 + \frac{1}{x_n}$. Prove that the sequence converges and find its limit.

Proof. First, we note that $x_n \geq 2\,\forall\, n$. Now, we aim to show that (x_n) is contractive with $K = \frac{1}{4}$:

$$x_{n+2} - x_{n+1} = 2 + \frac{1}{x_{n+1}} - \left(2 + \frac{1}{x_n}\right) = \frac{1}{x_{n+1}} - \frac{1}{x_n} = \frac{x_n - x_{n+1}}{x_{n+1} \cdot x_n}$$

$$\implies |x_{n+2} - x_{n+1}| = \frac{|x_n - x_{n+1}|}{x_n \cdot x_{n+1}}$$

$$x_n, x_{n+1} \geq 2 \implies x_n \cdot x_n \cdot x_{n+1} \geq 4$$

$$\implies \forall n \geq 1, |x_{n+2} - x_{n+1}| \leq \frac{1}{4}|x_{n+1} - x_n|$$

$$\overset{\text{theorem 2.12}}{\implies} (x_n) \text{ contractive, hence convergent}$$

We can now find the limit using the recursive definition; let $X = \lim_{n\to\infty} x_n$. $x_n \geq 2$, in particular, it is $\neq 0$ for any n. Then, we have:

$$X = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \left(2 + \frac{1}{x_n}\right) = 2 + \frac{1}{x} = X$$

$$\implies X = 2 + \frac{1}{X} \implies X^2 - 2X - 1 = 0$$

$$\implies X = 1 \pm \sqrt{2}$$

$1 - \sqrt{2} < 0$, which can't hold since $x_n \geq 0 \, \forall \, n$, hence it must be that $X = 1 + \sqrt{2}$. ■

⊛ **Example 2.17**

Show that the sequence $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}, n \geq 1$, diverges.

Proof. Note that

$$x_{2n} - x_n = \underbrace{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}}_{n \text{ terms, each } \geq \frac{1}{2n}} \geq n \cdot \frac{1}{2n} \geq \frac{1}{2}, \quad *$$

which means that the sequence cannot be Cauchy hence it cannot be convergent (see theorem 2.9).

More thoroughly, suppose (x_n) is convergent, that is, it is Cauchy. Take $\varepsilon = \frac{1}{4}$; since (x_n) Cauchy, there must exist some N such that $\forall \, n, m \geq N$,

$$|x_n - x_m| < \varepsilon = \frac{1}{4}.$$

But if we take, then, $n = 2N$ and $m = N$, then

$$|x_{2N} - x_N| < \frac{1}{4},$$

which is impossible, as we have shown in $*$ that $|x_{2N} - x_N| \geq \frac{1}{2}\ \forall N$, hence we have reached a contradiction. ■

2.7 Euler's Number e

Remark 2.25. *In the following section, we consider the sequences*

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

and

$$y_n = \left(1 + \frac{1}{n}\right)^{1+n}.$$

We consider the following propositions regarding the sequences.

↪ **Proposition 2.8: Step 1**

x_n is strictly increasing. [45]

↪ **Proposition 2.9: Step 2**

y_n is strictly decreasing.[46]

↪ **Proposition 2.10: Step 3**

For any n, k, $x_n < y_k$. [47]

↪ **Proposition 2.11: Step 4**

(x_n) is bounded from above and (y_n) is bounded from below. [48]

↪ **Proposition 2.12: Step 5**

(x_n) and (y_n) are converging sequences that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n,$$

which we denote by the number e.[49]

[45] Proof sketch: lots of very ugly algebra, starring Bernoulli's inequality.

[46] Proof sketch: precisely the same idea as Step 1, with just ast much ugly algebra.

[47] Proof sketch: very a-lá Nested Interval Property. 3 cases.

[48] Proof sketch: follows directly from Step 3; any element of y_n, namely y_1, upper bounds x_n, and any element of x_n, namely x_1, lower bounds y_n.

[49] Proof sketch: the sequences converge by MCT (following from the previous steps). y_n is just x_n times a term (note

Remark 2.26. *Step 3, Step 4, Step 5 are "easier"; the main parts of the proof deal with Step 1, Step 2. We will prove it using Bernoulli's Inequality.*

↪ Proposition 2.13: Bernoulli's Inequality

For all $x > -1$ and all $n \in \mathbb{N}$,

$$(1+x)^n \geq 1 + nx$$

Proof. We proceed by induction; fixing $x > -1$, let $S \subseteq \mathbb{N}$ the set for which the inequality holds. $n = 1 \implies (1+x)^1 \geq 1 + x$, which clearly holds, ie $1 \in S$. Suppose $n \in S$, that is,

$$(1+x)^n \geq 1 + nx$$

holds. Since $1 + x > 0$, we can multiply both sides by $1 + x$:

$$(1+x)^n \cdot (1+x) = (1+x)^{n+1} \geq (1+nx)(1+x) = 1 + nx + x + \overbrace{nx^2}^{\geq 0} \geq 1 + (n+1)x$$
$$\implies n + 1 \in S$$

Hence, by the axiom of induction, $S = \mathbb{N}$. ■

Proof. (Of Step 1) We will show that $\frac{x_{n+1}}{x_n} > 1 \,\forall\, n \in \mathbb{N}$. From our definition, we have

$$\begin{aligned}
\frac{x_{n+1}}{x_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} &= \frac{n+2}{n+1} \cdot \frac{(n+2)^n n^n}{[(n+1)^2]^n} \\
&= \frac{n+2}{n+1}\left[\frac{n^2+2n}{n^2+2n+1}\right]^n \\
&= \frac{n+2}{n+1}\left[\frac{n^2+2n+1-1}{n^2+2n+1}\right]^n \\
&= \frac{n+2}{n+1}\left[1 - \frac{1}{n^2+2n+1}\right]^n \\
&= \frac{n+2}{n+1}\left[1 - \frac{1}{(n+1)^2}\right]^n
\end{aligned}$$

By Bernoulli's Inequality with $x = -\frac{1}{(n+1)^2} > -1$, we have that

$$\left(1 - \frac{1}{(n+1)^2}\right)^n \geq 1 - \frac{n}{(n+1)^2},$$

which gives with our results above

$$\begin{aligned}\frac{x_{n+1}}{x_n} \geq \frac{n+2}{n+1}\left(1-\frac{n}{(n+1)^2}\right) &= \frac{n+2}{n+1}\cdot\frac{n^2+n+1}{(n+1)^2} \\ &= \frac{n^3+n^2+n+2n^2+2n+2}{n^3+3n^2+3n+1} \\ &= \frac{n^3+3n^2+3n+\mathbf{2}}{n^3+3n^2+3n+1} \\ &= \frac{n^3+3n^2+3n+1}{n^3+3n^2+3n+1}+\frac{1}{n^3+3n^2+3n+1} \\ &= 1+\frac{1}{n^3+3n^2+3n+1} > 1\end{aligned}$$

hence, $\frac{x_{n+1}}{x_n} > 1 \implies x_{n+1} > x_n \,\forall\, n$, ie it is strictly increasing. ■

Proof. (Of Step 2) We need to show $\frac{y_n}{y_{n+1}} > 1 \,\forall\, n > 1$. We have

$$\begin{aligned}\frac{y_n}{y_{n+1}} &= \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n+1}\right)^{n+2}} = \frac{\left(\frac{n+1}{n}\right)^{n+1}}{\left(\frac{n+2}{n+1}\right)^{n+2}} = \frac{n+1}{n+2}\cdot\frac{\frac{(n+1)^{n+1}}{n^{n+1}}}{\frac{(n+2)^{n+1}}{(n+1)^{n+1}}} \\ &= \frac{n+1}{n+2}\cdot\frac{\left[(n+1)^2\right]^{n+1}}{n^{n+1}(n+2)^{n+1}} = \frac{n+1}{n+2}\left[\frac{n^2+2n+1}{n^2+2n}\right]^{n+1} \\ &= \frac{n+1}{n+2}\cdot\left[1+\frac{1}{n^2+2n}\right]^{n+1} \\ \text{Bernoulli's Inequality}\, x = \frac{1}{n^2+2n} \implies \frac{y_n}{y_{n+1}} &\geq \frac{n+1}{n+2}\left[1+\frac{n+1}{n^2+2n}\right] \\ &= \frac{n+1}{n+2}\cdot\frac{n^2+3n+1}{n^2+2n} \\ &= \frac{n^3+3n^2+n+n^2+3n+1}{n^3+2n^2+2n^2+4n} = \frac{n^3+4n^2+4n+\mathbf{1}}{n^3+4n^2+4n} \\ &= 1+\frac{1}{n^3+4n^2+4n} > 1\end{aligned}$$

Hence, $\forall\, n$, $\frac{y_n}{y_{n+1}} > 1 \implies y_n > y_{n+1}$, ie, it is strictly decreasing. ■

Proof. (Step 3) We aim to show that for all n, k, $x_n < y_k$.

- (Case 1) $n = k$:

$$x_n = \left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{n}\right)^{n+1} = y_n$$

- (Case 2) $n > k$:

$$y_k > y_n > x_n \text{ by Case 1, since } (y_n) \text{ strictly decreasing.}$$

- (Case 3) $n < k$:

$$x_n < x_k < y_k \text{ by Case 1, since } (x_n) \text{ strictly increasing.}$$

■

Proof. (Of Step 4) Since $x_n < y_k \,\forall\, k, n$, we have that

$$x_n < y_1 = 4 \,\forall\, n,$$

and

$$2 = x_1 < y_k \,\forall\, k,$$

hence (x_n) is bounded from above (by y_1, say) and (y_n) is bounded from below (by x_1, say). ■

Proof. (Of Step 5) Since (x_n) increasing and bounded from above, it is converging by Monotone Convergence Theorem. Similarly, (y_n) is decreasing and bounded from below, hence it too converges. We have too that

$$y_n = \left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n}\right)^{n} = \left(1 + \frac{1}{n}\right) x_n$$

Since $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 1$, we have, from proposition 2.1, that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) \cdot \lim_{n\to\infty} x_n = \lim x_n,$$

that is, (x_n) and (y_n) converge to the same limit, which we define as

$$e \equiv \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n} = \left(1 + \frac{1}{n}\right)^{n+1}.$$

■

Remark 2.27. *This proof naturally gives that* $\forall\, n \in \mathbb{N}$,

$$\left(1 + \frac{1}{n}\right)^{n} < e < \left(1 + \frac{1}{n}\right)^{n+1},$$

which we can use to estimate e *arbitrarily.*

⊛ **Example 2.18**

Consider the sequence $S_n = \sum_{k=0}^{n} \frac{1}{k!}$. Show that the sequence (S_n) is Cauchy and that $\lim_{n\to\infty} S_n = e$.

2.8 Limit Points

↪ **Definition 2.12: Limit Point**

Let (x_n) be a sequence. A number $x \in \mathbb{R}$ is called a *limit point* or *accumulation point* if $\exists$ a subsequence (x_{n_k}) of (x_n) such that $\lim_{k\to\infty} x_{n_k} = x$. We denote by $\mathscr{L}$ the set of limit points.

Remark 2.28. *Note that $\mathscr{L}$ could be an empty set; however, if x_n bounded, then $\mathscr{L} \neq 0$ by* Bolzano-Weirestrass Theorem. *Further, $\mathscr{L}$ is a bounded subset of $\mathbb{R}$.*

↪ **Proposition 2.14**

Let (x_n) be a sequence. Then, $x \in \mathscr{L}$ iff $\forall\, \varepsilon > 0$ the set $\{n : |x_n - x| < \varepsilon\}$ is infinite.

Proof. Suppose first that $x \in \mathscr{L}$ and let (x_{n_k}) be a subsequence such that $\lim_{k\to\infty} x_{n_k} = x$. Let $\varepsilon > 0$. Then $\exists K \in \mathbb{N}$ s.t. $\forall\, k \geq K$, $|x_{n_k} - x| < \varepsilon$.

Then, the set

$$\{n_k : k \geq K\} \subseteq \{n : |x_n - x| < \varepsilon\}.$$

Since the LHS set is infinite, the RHS must be too.

We now show the converse. Suppose that $\forall\, \varepsilon > 0$, the set $\{n : |x_n - x| < \varepsilon\}$ is infinite. Take $\varepsilon = 1$, then the set $\{n : |x_n - x| < 1\}$ is an infinite set. Take $n_1 = \min\{n : |x_n - x| < 1\}$. We can now define n_k, where $k = 2, 3, \ldots$ recursively. Suppose that some n_k chosen. Then, take $\varepsilon = \frac{1}{k}$, and consider

$$\{n : n > n_k, |x_n - x| < \frac{1}{k}\}.$$

This set has infinitely many elements, since $\{n : |x_n - x| < \frac{1}{k}\}$ is also infinite. We then set $n_{k+1} = \min\{n : n > n_k, |x_n - x| < \frac{1}{k}\}$, which defines a strictly increasing sequence $(n_k)_{k\geq 1}$ of natural numbers such that for any k, $|x_{n_k} - x| < \frac{1}{k}$. So, $\lim_{k\to\infty} |x_{n_k} - x| = 0$ which gives that $\lim_{k\to\infty} x_{n_k} = x$, so $x \in \mathscr{L}$. ■

↪ **Theorem 2.13**

Let (x_n) be a bounded sequence. Then,

1. $\overline{\lim}_{n\to\infty} x_n = \sup \mathscr{L}$
2. $\underline{\lim}_{n\to\infty} x_n = \inf \mathscr{L}$

Remark 2.29. *The following proof shows even more, that is,* $\overline{\lim}_{n\to\infty} x_n = \sup \mathscr{L}$ *and* $\overline{\lim}_{n\to\infty} x_n \in \mathscr{L}$ *(same for* $\underline{\lim}_{n\to\infty}$*).*

Remark 2.30. *1.* $\implies$ *2. by changing the sign of* x_n*, as "always".*

Proof. We[50] will show first that $\overline{\lim}_{n\to\infty} x_n \geq \sup \mathscr{L}$.

Let $x \in \mathscr{L}$ and let (x_{n_k}) be a subsequence such that $\lim_{k\to\infty} x_{n_k} = x$. Let $y_n = \sup\{x_m : m \geq n\}$. We have, then, that $y_n \geq x_n$, and that $\overline{\lim}_{n\to\infty} x_n = \lim y_n$, hence $\forall\, k$, $y_{n_k} \geq x_{n_k}$, and that $\overline{\lim}_{n\to\infty} x_n = \lim_{k\to\infty} y_{n_k}$ ((y_n) is a convergent sequence, so any subsequence converges to the same limit.) So, we have that

$$\overline{\lim}_{n\to\infty} x_n = \lim_{k\to\infty} y_{n_k} \geq \lim_{k\to\infty} x_{n_k} = x,$$

that is, for any $x \in \mathscr{L}$, $\overline{\lim}_{n\to\infty} x_n \geq x \implies \overline{\lim}_{n\to\infty} x_n$ upper bounds $\mathscr{L}$. Since sup least upper bound, it must be that $\overline{\lim}_{n\to\infty} x_n \geq \sup \mathscr{L}$.

We now show that $\overline{\lim}_{n\to\infty} x_n \leq \sup \mathscr{L}$; indeed, we will show that $\overline{\lim}_{n\to\infty} x_n \in \mathscr{L}$ and thus must be $\leq \sup \mathscr{L}$. We will show this by constructing a subsequence of (x_n) that converges to $\overline{\lim}_{n\to\infty} x_n$.

Set $n_1 = 1$. Suppose n_k defined. Then,

$$y_{n_k+1} = \sup\{x_n : n \geq n_k + 1\}.$$

If we consider $y_{n_k+1} - \frac{1}{k+1}$, then this number is smaller than y_{n_k+1} and is thus not an upper bound for the set $\{x_n : n \geq n_k + 1\}$. Then there exists some $n_{k+1} \geq n_k + 1 > n_k$ such that

$$y_{n_k+1} - \frac{1}{k+1} \leq x_{n_{k+1}} \leq y_{n_{k+1}} = \sup\{x_n : n \geq n_{k+1}\}.$$

So, we have constructed a strictly increasing sequence (n_k) of natural numbers such that $\forall\, k \geq 1$,

$$y_{n_k+1} - \frac{1}{k+1} \leq x_{n_{k+1}} \leq y_{n_{k+1}}. \quad \star$$

[50] Sketch: show double inequality. First, show that $\limsup \geq \sup \mathscr{L}$, by using the fact that x_n bounded, and so y_n (the sequence that converges to $\limsup$) converges and is $\geq x_n\ \forall\, n$, and so must be greater than any subsequence. To show $\limsup \leq \sup \mathscr{L}$, show that $\limsup \in \mathscr{L}$, and hence must be equal to $\sup \mathscr{L}$. To do this, create to two subsequences of y_n (note - y_n NOT x_n) that both converge to x_n. Note that these exist since y_n converges. The real proof is in constructing the sequence of indices n_k such that y_{n_k} "bounds" (so to speak) some x_{n_k}. Then, using squeeze theorem, $x_{n_k} \to \limsup x_n$, so $\limsup x_n \in \mathscr{L}$ and the proof is complete.

Consider the subsequences (y_{n_k+1}) and $(y_{n_{k+1}})$ of (y_n), and a subsequence $(x_{n_{k+1}})$ of (x_n). Since y_n converges, and $\lim_{n\to\infty} y_n = \overline{\lim}_{n\to\infty} x_n$, we have that

$$\lim_{k\to\infty} y_{n_k+1} = \overline{\lim}_{n\to\infty} x_n; \text{ and } \lim_{k\to\infty} y_{n_{k+1}} = \overline{\lim}_{n\to\infty} x_n,$$

and so, given this and $\star$, by the The Squeeze Theorem, $\lim_{k\to\infty} x_{n_{k+1}} = \overline{\lim}_{n\to\infty} x_n$, and so $\overline{\lim}_{n\to\infty} x_n \in \mathscr{L}$. ■

↪ **Corollary 2.3**

Let (x_n) be a bounded sequence and $\alpha = \underline{\lim}_{n\to\infty} x_n, \beta = \overline{\lim}_{n\to\infty} x_n$. Then, $\alpha, \beta \in \mathscr{L}$ (that is, they are limit points of (x_n)), and for any $x \in \mathscr{L}$, $\alpha \leq x \leq \beta$ (that is, $\mathscr{L}$ is a closed set).

2.9 Properly Divergent Sequences

↪ **Definition 2.13: Properly Divergent Sequences**

Let (x_n) be a sequence. We say that (x_n) *properly diverges* to ∞ if for any $\mathbb{R}\ \exists N \in \mathbb{N}$ such that $\forall\, n \geq N, x_n \geq \alpha$. We write

$$\lim_{n\to\infty} x_n = \infty.$$

That is,

$$(\forall\, \alpha \in \mathbb{R})(\exists N \in \mathbb{N})(\forall\, n \geq N)(x_n \geq \alpha).$$

We analogously say (x_n) diverges to $-\infty$ if $\forall\, \alpha \in \mathbb{R} \exists N \in \mathbb{N}$ s.t. $\forall\, n \geq N, x_n \leq \alpha$.

⊛ **Example 2.19**

$x_n = n$ properly diverges to ∞; $x_n = -n$ properly diverges to $-\infty$.

⊛ **Example 2.20**

Let $C > 1$. Then, $\lim_{n\to\infty} C^n = \infty$.

Proof. Write $C = 1 + x$ where $x > 0$. By Bernoulli's Inequality, $\forall\, n \geq 1$,

$$C^n = (1+x)^n \geq 1 + nx.$$

Let $\alpha \in \mathbb{R}$. If $\alpha \leq 0$, then $\forall n, C^n > \alpha$. If $\alpha > 0$, let $N \in \mathbb{N}, N \geq \frac{\alpha}{x}$. SO, $\forall n \geq N$, $C^n \geq 1 + nx > \alpha$ and $\lim_{n\to\infty} C^n = \infty$. ■

↪ **Proposition 2.15**

Let (x_n) be increasing. Then $\lim_{n\to\infty} x_n = \infty$ iff x_n not bounded from above.

Proof. ($\implies$) Let $M \in \mathbb{R}$. Since $(x_n) \to \infty$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N, x_n \geq M$, that is, x_n is unbounded.

($\impliedby$) Suppose x_n not bounded from above. Let $\alpha \in \mathbb{R}$, then, $\exists N$ s.t. $x_N > \alpha$. (x_n) increasing $\implies \forall n \geq N, x_n \geq x_N > \alpha \implies \lim_{n\to\infty} x_n = \infty$. ■

↪ **Proposition 2.16**

Let (x_n) be decreasing. Then $\lim_{n\to\infty} x_n = -\infty \iff (x_n)$ not bounded from below.

Remark 2.31. *Follows from proposition 2.15.*

↪ **Proposition 2.17**

Let $(x_n), (y_n)$ be sequences such that $x_n \leq y_n \forall n$. Then

1. $\lim_{n\to\infty} x_n = \infty \implies \lim_{n\to\infty} y_n = \infty$
2. $\lim_{n\to\infty} y_n = -\infty \implies \lim_{n\to\infty} x_n = -\infty$

Proof. (Of 1.) Let $\alpha \in \mathbb{R}$; since $\lim x_n = \infty, \exists N$ s.t. $\forall n \geq N, x_n \geq \alpha \implies \forall n \geq y_n \geq x_n \geq \alpha \implies \lim y_n = \infty$. ■

↪ **Proposition 2.18**

Let (x_n) be a sequence and $c > 0$. Then $\lim x_n = \infty \iff \lim c \cdot x_n = \infty$. The converse follows for $c < 0$ and $\to -\infty$.

Proof. ■

↪ **Proposition 2.19**

Let (x_n) and (y_n) be strictly positive sequences. Suppose that for some $L > 0$,

$$\lim_{n\to\infty} \frac{x_n}{y_n} = L.$$

Then, $\lim x_n = \infty \iff \lim y_n = \infty$.

Proof. Take $\varepsilon = \frac{L}{2}$. Then, $\frac{x_n}{y_n} \to L \implies \exists N$ s.t. $\forall n \geq N, L - \varepsilon < \frac{x_n}{y_n} < L + \varepsilon \iff L - \frac{L}{2} < \frac{x_n}{y_n} < L + \frac{L}{2}$. So, $\forall n \geq N, \frac{L}{2} < \frac{x_n}{y_n} < \frac{3L}{2} \implies \frac{L}{2} y_n < x_n < \frac{3}{2} L y_n$. Hence, if $x_n \to \infty$, it must be that $y_n \to \infty$, by the previous inequality. The other side of the implication follows similarly. ■

↪ **Proposition 2.20**

Let $(x_n), (y_n)$ be two sequences such that (x_n) is properly divergent and y_n bounded. Then their sum is also diverging.

Proof. ■

⊛ **Example 2.21**

$x_n = n, y_n = \frac{-n}{2}$. $x_n \to \infty, y_n \to -\infty$ while $x_n + y_n \to \infty$.

↪ **Definition 2.14: Limsup (Generalized)**

Let (x_n) be a sequence bounded from *above*. Define, as previously, $y_n := \sup\{x_k : k \geq n\}$; recall that this sequence is decreasing, and moreover, that $\lim_{n\to\infty} y_n$ exists.

This limit is finite, as seen previously, if y_n bounded from *below*. If it is not, y_n diverges and as it is decreasing, $\lim y_n = -\infty$. Recall that $\limsup x_n = \lim y_n$, hence if x_n bounded from above, $\overline{\lim}_{n\to\infty} x_n$ exists, and is either a real number, or $-\infty$.

In the case x_n *not bounded above*, then we *define* $\overline{\lim}_{n\to\infty} x_n = \infty$. In this way, we define $\overline{\lim}_{n\to\infty} x_n$ for all sequences, regardless of convergence or boundedness.

↪ **Definition 2.15: Liminf (Generalized)**

Let (x_n) be a sequence. If (x_n) bounded from below, let $z_n = \inf\{x_k : k \geq n\}$. This is an increasing sequence. We define $\underline{\lim}_{n\to\infty} x_n := \lim z_n$; $\lim z_n$ finite $\iff$ x_n bounded from above, and infinite otherwise (ie, $\underline{\lim}_{n\to\infty} x_n = \infty \iff x_n$ unbounded from above).

If x_n not bounded from below, then $\underline{\lim}_{n\to\infty} x_n := -\infty$.

↪ **Proposition 2.21**

Practically all previously proven properties of limsup/liminf hold with these generalizations:

1. $\underline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} x_n, -\infty < x < \infty \, \forall \, x \in \mathbb{R}$.

2. (x_n) converging or properly diverging and $\lim x_n = a \iff \underline{\lim}_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n = a$ (noting that $a \in \mathbb{R} \cup \{-\infty, \infty\}$[51])

3. $\underline{\lim}_{n\to\infty}(-x_n) = -\underline{\lim}_{n\to\infty} x_n$[52]

4. $\overline{\lim}_{n\to\infty} x_n = \inf\{t : \{n : x_n > t\}$ finite or empty$\}$ and $\underline{\lim}_{n\to\infty} x_n = \sup\{t : \{x_n < t\}$ finite or empty$\}$.[53]

[53]See: Extended Real Line

[53]We take, here, $-(-\infty) \equiv \infty$

[53]We *define* $\inf \varnothing = \infty$ and $\sup \varnothing = -\infty$, as a convention. Moreover, if a set A is not bounded from below, then we have $\inf A = -\infty$, and if A not bounded from above, $\sup A = \infty$.

↪ **Definition 2.16: Limit Set**

The *limit set* of a sequence (x_n) is the collection of all $x \in \mathbb{R} \cup \{-\infty, \infty\}$ s.t. for some subsequence (x_{n_k}) of x_n, $\lim_{k\to\infty} x_{n_k} = x$. Then, we have, as before, $\overline{\lim}_{n\to\infty} x_n = \sup \mathscr{L}$, $\underline{\lim}_{n\to\infty} x_n = \inf \mathscr{L}$.

Remark 2.32. *Not all concepts defined on convergent/bounded sequences extend easily to properly divergent sequences. For instance,* $\overline{\lim}_{n\to\infty}(x_n + y_n) \leq \overline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n$ *holds for bounded sequences* x_n, y_n*, but does not generally hold if* $\overline{\lim}_{n\to\infty} x_n = \infty$*, etc..*

3 Functional Limits and Continuity

↪ **Definition 3.1: Cluster Point**

Let[54] $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is called a *cluster* or *limit* point of A if $\forall \, \varepsilon > 0$, $\exists x \in A$, $x \neq c$, s.t. $0 < |x - c| < \varepsilon$.

[54]Read: a point is a cluster point if there exists points (other than itself) in the set that are arbitrarily close ("epsilon close") to it.

Remark 3.1. *Note that this definition does* not *require* $c \in A$.

↪ **Proposition 3.1**

Let $A \subseteq \mathbb{R}$, $c \in \mathbb{R}$. Then, TFAE:

1. c is a cluster point of A

2. $\exists$ a sequence (x_n) s.t. $\forall \, x_n \in A, x_n \neq c$, and $\lim x_n = c$.

Proof. (1. $\implies$ 2.) Let c be a cluster point of A, and take $\varepsilon = \frac{1}{n}$ in the definition of a cluster point. Then, by definition, $\exists x_n \in A, x_n \neq c,$ s.t. $0 < |x_n - c| < \frac{1}{n}$. This defines a sequence $x_n \in A$, $x_n \neq c \, \forall \, n$, with the property that $\forall \, n, |x_n - c| < \frac{1}{n}$. Moreover, this gives, by definition, that $\lim x_n = c$.

(2. $\implies$ 1.) Suppose there exists a sequence (x_n) in A, $x_n \neq c$, such that $\lim x_n = c$. Take $\varepsilon > 0$, and let N be such that $\forall n \geq N$, $|x_n - c| < \varepsilon$. Take $x = x_N$; then, we have that $x \in A, x \neq c$, and $0 < |x - c| < \varepsilon$. By definition, then, c is a cluster point, and the proof is complete. ■

⊛ **Example 3.1**

Let $A = (0, 1)$. Then, 0 is a cluster point of A.

Proof. Consider the sequence $x_n = \frac{1}{n+1}$. Then, since $0 < (x_n) < 1$, $x_n \in A \,\forall\, n$, moreover, $x_n \neq 0$. Hence, $\lim x_n = 0$, hence 0 is a cluster point of A. ■

⊛ **Example 3.2**

Let $A = (0, 1) \cup \{5\}$. Is 5 a cluster point?

Proof. No; it is impossible to find arbitrarily (ε) close points to 5 in the set; $\nexists x \in A, x \neq 5$ such that $0 < |x - 5| < \varepsilon$. Then, the set of all cluster points of A is equal $[0, 1]$. ■

⊛ **Example 3.3**

Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then, $c = 0$ is the only cluster point of A.

Proof. We show first $c = 0$ is indeed a cluster point. Let $x_n = \frac{1}{n}$; then, $x_n \in A \,\forall\, n, x_n \neq 0$, and moreover, $\lim x_n = 0$, hence $c = 0$ a cluster point of A.

We now show that 0 is the only cluster point of A. ■

⊛ **Example 3.4**

Let $A = \mathbb{Q}$. Then, the set of cluster points is precisely $\mathbb{R}$.

Proof. Take $x \in \mathbb{R}, \varepsilon > 0$. Consider the interval $(x, x + \varepsilon)$; by density of the rationals, $\exists q \in \mathbb{Q}$ s.t. $q \in (x, x + \varepsilon)$. Hence, $\exists q \in \mathbb{Q}, q \neq x$ s.t. $0 < |x - q| < \varepsilon$, hence, x a cluster point of A. ■

↪ **Definition 3.2: Functional Limits**

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and c a cluster point of A. Then, we say that the limit of f at c is L, denoted

$$\lim_{x \to c} f(x) = L,$$

if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$ satisfying $0 < |x - c| < \delta$, we have that $|f(x) - L| < \varepsilon$.

Remark 3.2. *"As x gets closer and closer to c, $f(x)$ gets closer and closer to L".*

Remark 3.3. *The point c may or may not be in A (for instance, $\lim_{x\to 0} \frac{\sin x}{x} = 1$). However, it must be that c is a cluster point of A; this is what "allows" the arbitrary closeness to L in the definition of a limit.*

Remark 3.4. *This definition is often called the "$\varepsilon - \delta$" definition of functional limits. Quantified, it states*

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)(0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon).$$

⊛ **Example 3.5**

Let $A = (0, \infty)$, let $f(x) = \frac{1}{x}$, $x \in A$, and let $c \in A$. Prove that $\lim_{x\to c} f(x) = \frac{1}{c}$.

Proof. Note: c a cluster point of A since for $\varepsilon > 0, x = c + \frac{\varepsilon}{2} \in A, x \neq c, 0 < |x - c| = \frac{\varepsilon}{2} < \varepsilon$ (hence the limit is indeed well-defined).

Fix $\varepsilon > 0$; take $\delta = \min\{\frac{1}{2}c, \frac{1}{2}c^2\varepsilon\}$. Then,

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{c - x}{xc}\right| = \frac{|x - c|}{|xc|} < \frac{\delta}{|xc|},$$

if $x \in A$ is such that $0 < |x - c| < \delta$. Since $|x - c| < \delta$, we have that $x - c > -\delta \implies x > c - \delta$. We also have, by definition, $\delta \leq \frac{1}{2}c$, hence, $x > \frac{c}{2}$. This gives that $\frac{1}{|xc|} = \frac{1}{xc} < \frac{1}{\frac{c}{2}c} = \frac{2}{c^2}$. We thus have that, for $0 < |x - c| < \delta$, that $\left|\frac{1}{x} - \frac{1}{c}\right| < \frac{2}{c^2}\delta$. But we also have that $\delta \leq \frac{c^2}{2}\varepsilon$, hence $\left|\frac{1}{x} - \frac{1}{c}\right| < \frac{2}{c^2}\frac{c^2}{2}\varepsilon \implies \left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon$. Thus, $\lim_{x\to c} \frac{1}{x} = \frac{1}{c}$. ■

3.1 Sequential Characterization of Functional Limits

↪ **Theorem 3.1**

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and let c be a cluster point of A. Then, TFAE:

1. $\lim_{x\to c} f(x) = L$.
2. For any sequence $(x_n) \in A$, $x_n \neq c$, such that $\lim x_n = c$, we have that the sequence $(f(x_n))$ converges to L, that is, $\lim_{n\to\infty} f(x_n) = L$.

Proof. (1. $\implies$ 2.) By 1., $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A, x \neq c$ such that $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$. Let (x_n) be a sequence in A s.t. $x_n \neq c$ and $\lim_{n\to\infty} x_n = c$. We wish to show that $\lim_{n\to\infty} f(x_n) = L$. Take $\varepsilon > 0$; then, we have that $\exists \delta > 0$ s.t. $\forall x \in A$ satisfying $0 < |x - c| < \delta$, we have $|f(x) - L| < \varepsilon$. Fix such a δ; then, since $\lim_{n\to\infty} x_n = c$, $\exists N$ s.t. $\forall n \geq N, |x_n - c| < \delta$. Then, it follows from the definition of δ that for $n \geq N$, $|f(x_n) - L| < \varepsilon$, hence $\lim_{n\to\infty} f(x_n) = L$, and 2. holds.

(2. $\implies$ 1.) Suppose not. Then, $\forall \varepsilon > 0$, we can find $\delta > 0$ such that $\forall x \in A$ s.t. $0 < |x - c| < \delta$ we have $|f(x) - L| < \varepsilon$. But then, this means that $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0, \exists x \in A, x \neq C$ s.t. $0 < |x - c| < \delta$ and $|f(x) - L| \geq \varepsilon_0$. So, for this $\varepsilon_0 > 0$, we can take $\delta = \frac{1}{n}$, which gives $x_n \in A, x_n \neq c$, such that $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \varepsilon_0$. This gives us a sequence $(x_n) \in A, x_n \neq c$, such that $\lim_{n\to\infty} |x_n - c| = 0 \implies \lim_{n\to\infty} x_n = c$, and $|f(x_n) - L| \geq \varepsilon_0 \, \forall n$. But this means that we have a sequence x_n s.t. $\lim x_n = c$, and the sequence $(f(x_n))$ does not converge to L. But this contradicts 2.; hence, we have come to a contradiction, and 1. holds. ■

↪ Proposition 3.2

A functional limit is unique. That is, if $f : A \to \mathbb{R}$ and c a cluster point of A, if $\lim_{x\to c} f = L$ an $\lim_{x\to c} f = M$, $L = M$.

Proof. (Sequential) Let x_n be a sequence in A such that $x_n \neq c$ and $\lim_{n\to\infty} x_n = c$. Then by the sequential characterization, $\lim_{x\to c} f(x) = L \implies \lim_{n\to\infty} f(x_n) = L$, and $\lim_{x\to c} f(x) = M \implies \lim_{n\to\infty} f(x_n) = M$. That is, the sequence $f(x_n)$ converges to both L and M, but the limit of a sequence is unique (if it exists), hence $L = M$. ■

$(\varepsilon - \delta)$[55] Take $\varepsilon = \frac{|L-M|}{2}$, and suppose $L \neq M$, hence $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = L, \exists \delta_1 > 0$ s.t. $\forall x \in A 0 < |x - c| < \delta_1$, we have $|f(x) - L| < \varepsilon$. Similarly, since $\lim_{x\to c} f(x) = M, \exists \delta_2 > 0$ s.t. $\forall x \in A, 0 < |x - c| < \delta_2$, we have that $|f(x) - M| < \varepsilon$. Take $\delta = \min\{\delta_1, \delta_2\}$ and let $x \in A$ s.t. $0 < |x - c| < \delta$. Then,

$$\begin{aligned} |L - M| = |L - f(x) + f(x) - M| &\leq |L - f(x)| + |f(x) - M| \\ &< \varepsilon + \varepsilon = 2\varepsilon = |L - M|, \end{aligned}$$

which implies $|L - M| < |L - M|$, a contradiction. Hence, $L = M$. ■

[55] Note the similarity of this proof and that which we used to prove limits of sequences are unique (theorem 2.1).

↪ Theorem 3.2: Algebraic Properties of Functional Limits

Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, and let c be a cluster point of A. Suppose $\lim_{x\to c} f(x) = L$

and $\lim_{x\to c} g(x) = M$. Then,

1. For any constant $k \in \mathbb{R}$, $\lim_{x\to c}(k \cdot f(x)) = k \cdot \lim_{x\to c} f(x) = k \cdot L$.
2. $\lim_{x\to c}(f(x) + g(x)) = L + M$
3. $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
4. If $g(x) \neq 0 \,\forall\, x \in A$, and $M \neq 0$, $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$.

Proof. (Of 3.; Sequential) Let x_n be sequence in A, $x_n \neq c$, and $\lim_{n\to\infty} x_n = c$. Then, $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x) = M$. But then, by product rule of converging sequences (proposition 2.1), $\lim_{n\to\infty}(f(x_n)g(x_n)) = \lim_{n\to\infty} f(x_n) \lim_{n\to\infty} g(x_n) = L \cdot M$. Moreover, by sequential characterization of functional limits, we have that $\lim_{x\to c}(f(x)g(x)) = L \cdot M$. ■

($\varepsilon - \delta$) Since $\lim_{x\to c} f(x) = L$, if we take $\varepsilon = 1$, we can find $\delta_1 > 0$ s.t. $\forall\, x \in A$, $x \neq c, 0 < |x - c| < \delta_1$, we have that $|f(x) - L| < 1$. For such an x, we have that $|f(x)| = |f(x) - L + L| \leq |f(x) - L| + |L| < 1 + L$. Take now $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, we can find $\delta_2 > 0$ s.t. $\forall\, 0 < |x - c| < \delta_2$, we have that $|f(x) - L| < \frac{\varepsilon}{2(|M|+1)}$, $|g(x) - M| < \frac{\varepsilon}{2(|L|+1)}$. Take now $\delta = \min\{\delta_1, \delta_2\}$, and let x be s.t. $0 < |x - c| < \delta$. Then,

$$\begin{aligned} |(f(x) \cdot g(x)) - (L \cdot M)| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &\leq |f(x)g(x) - f(x)M| + |f(x)M - LM| \\ &= |f(x)||g(x) - M| + |M||f(x) - L| \\ &< (1 + |L|)|g(x) - M| + |M||f(x) - L| \\ &< (1 + |L|)\frac{\varepsilon}{2(|L|+1)} + |M|\frac{\varepsilon}{2(|M|+1)} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

where the fourth line follows directly from $\delta \leq \delta_1$ as defined previously. ■

↪ **Theorem 3.3: Functional Squeeze Theorem**

Let $A \subseteq \mathbb{R}$, $f, g, h : A \to \mathbb{R}$, and let c be a cluster point of A. Suppose that for all $x \in A$, we have that $f(x) \leq g(x) \leq h(x)$, and that $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} g(x) = L$.

Proof. (Sequential) Let x_n be a sequence in $A, x_n \neq c$ such that $\lim_{n\to\infty} x_n = c$. Then, we have $\lim_{n\to\infty} f(x_n) = \lim_{x\to c} f(x) = L$ and similarly $\lim_{n\to\infty} h(x_n) = \lim_{x\to c} h(x) = L$.

Now, we have that $\forall n,\ f(x_n) \leq g(x_n) \leq h(x_n)$. By the squeeze theorem for sequences, $\lim_{n\to\infty} g(x_n) = L \quad \blacksquare$.

$(\varepsilon - \delta)$ Let $\varepsilon > 0$. Since $\lim_{x\to c} f(x) = L$, $\exists \delta_1 > 0$ s.t. $\forall x \in A$ s.t. $0 < |x - c| < \delta_1$ we have $|f(x) - L| < \varepsilon$. Since $\lim_{x\to c} h(x) = L$, we have that $\exists \delta_2 > 0$ s.t. $\forall x \in A$ s.t. $0 < |x - c| < \delta_2$ we have $|h(x) - L| < \varepsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$ and let x be such that $x \in A, 0 < |x - c| < \delta$. Then,

$$-\varepsilon < f(x) - L \leq g(x) - L \leq h(x) - L < \varepsilon,$$

thus, $|g(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$, hence $\lim_{x\to c} g(x) = L$. $\blacksquare$

Remark 3.5. *Note the similarity between the $\varepsilon - \delta$ proofs above and the proofs of corresponding properties for sequences.*

$\hookrightarrow$ **Definition 3.3: Divergence Criterion of a Function**

Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. The following criterion state that the limit of f at c does not exist:

1. Suppose there exists a sequence $x_n \in A, x_n \neq c,$ s.t. $\lim_{n\to\infty} x_n = c$, s.t. $\lim_{n\to\infty} f(x_n)$ does not exist. Then, $\lim_{x\to c} f(x)$ also does not exist.
2. Suppose there exist two sequences $x_n, y_n \in A, x_n, y_n \neq c,$ s.t. $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c$, and the limits $\lim_{n\to\infty} f(x_n)$ and $\lim_{n\to\infty} f(y_n)$ exist, but these two limits are different, then $\lim_{x\to c} f(x)$ does not exist.

⊛ **Example 3.6:** $f(x) = \sin \frac{1}{x}$

Let $A = (0, \infty)$, $f(x) = \sin \frac{1}{x}$ and $c = 0$. Then, $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist.

Proof. (Using divergence criterion 1.) Take $x_n = \frac{1}{(2n+1)\frac{\pi}{2}}$. Then, $x_n > 0$, and $\lim_{n\to\infty} x_n = 0 = c$. Moreover, $f(x_n) = \sin\left(\frac{1}{x_n}\right) = \sin\left((2n+1)(\frac{\pi}{2})\right) = (-1)^n$. This sequence does not converge, and so $\lim_{x\to 0} \sin \frac{1}{x}$ does not exist. $\blacksquare$

(Using divergence criterion 2.) Take $x_n = \frac{1}{2n\pi}, y_n = \frac{1}{2n\pi + \frac{\pi}{2}}$, noting that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$. Then, $f(x_n) = \sin \frac{1}{x_n} = \sin(2n\pi) = \forall n$, while $f(y_n) = \sin \frac{1}{y_n} = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1\,\forall n$, hence, $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, and thus $\lim_{x\to c} f(x)$ does not exist. $\blacksquare$

⊛ **Example 3.7: Abbott, 4.2E2**

Let $x_n = \frac{2}{\pi n}, y_n = \frac{1}{(2+n)\pi}$. Then, we have that both $(x_n) \to 0$ and $(y_n) \to 0$, but

$$f(x_n) = \cos\left(\frac{\pi n}{2}\right) = 0 \,\forall\, n; \quad f(y_n) = \cos((2+n)\pi) = 1 \,\forall\, n,$$

hence, $\lim_{n\to\infty} f(x_n) = 1 \neq \lim_{n\to\infty} f(y_n) = 0$, so the limit does not exist.

Consider now $\lim_{x\to 0} x \cos \frac{1}{x}$. Fix $\varepsilon > 0$, and take $\delta = \varepsilon$, then, we have that $\forall\, x$ s.t. $0 < |x - 0| < \delta$. Then, we have by properties of cos,

$$\left| x \cos \frac{1}{x} \right| \leq |x| < \delta = \varepsilon,$$

hence the function converges to 0.

⊛ **Example 3.8: Abbott, 4.2E14**

Let $f : A \subseteq \mathbb{R}, f : A \to \mathbb{R}, c \in \mathbb{R}$ be a cluster point of A, and $f(x) \geq 0 \,\forall\, x \in A$. Prove that $\lim_{x\to c} \sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)}$.

Proof. (Seq'n) Define $L := \lim_{x\to c} f(x)$. Then, we have that $\forall\, x_n \in A\backslash\{c\}$ s.t. $(x_n) \to c$, $\lim_{n\to\infty} f(x_n) = L$. We can write, then,

$$L = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sqrt{f(x_n)}\sqrt{f(x_n)} = \left(\lim_{n\to\infty} \sqrt{f(x_n)}\right)^2,$$

and taking the square root of both sides, we have the desired result. Note that this used the assumption that $\exists \lim_{n\to\infty} x_n \implies \exists \lim_{n\to\infty} \sqrt{x_n}$.

$(\varepsilon - \delta)$

■

3.2 Left/Right Limits

↪ **Definition 3.4: Left/Right Limits**

1. Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}$, and suppose that c is a cluster point of the set

$$A \cap (c, \infty) = \{x \in A : x > c\}.$$

Then we say that a real number L is the *right limit* of f at c, denoted

$$\lim_{x \to c^+} f(x) = L,$$

if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$ s.t. $0 < x - c < \delta \implies |f(x) - L| < \varepsilon$.

2. Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and suppose that c is a cluster point of

$$A \cap (-\infty, c) = \{x \in A : x < c\}.$$

Then we say that a real number L is the *left limit* of f at c, denoted

$$\lim_{x \to c^-} f(x) = L,$$

if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$ s.t. $-\delta < x - c < 0 \implies |f(x) - L| < \varepsilon$.

Remark 3.6. *Sometimes, but not always, the right/left endpoints are equivalent to the "usual" limit.*

⊛ **Example 3.9: The Heaviside Function**

Let $f : \mathbb{R} \to \mathbb{R}$ defined $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$. We have

$$\lim_{x \to 0^+} f(x) = 1; \quad \lim_{x \to 0^-} f(x) = 0.$$

Let $\varepsilon > 0$. Take $\delta > 0$. Then, $\forall x$ s.t. $0 < x < \delta, |f(x) - 1| = |1 - 1| = 0 < \varepsilon$, hence $\lim_{x \to 0^+} f(x) = 0$.

↪ **Proposition 3.3**

Let $A \subseteq \mathbb{R}, f : A \to \mathbb{R}$, and let c be a cluster point of the sets $A \cap (c, \infty)$ and $A \cap (-\infty, c)$. TFAE:

1. $\lim_{x \to c} f(x) = L$
2. $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$

Proof. (1. $\implies$ 2.) Let $\varepsilon > 0$ and δ s.t. $\forall x \in A$ s.t. $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$. Then, we have that $|f(x) - L| < \varepsilon \impliedby 0 < x - c < \delta$, and moreover, $|f(x) - L| < \varepsilon \impliedby -\delta < x - c < 0$, that is, $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$, hence 2. holds.

(2. $\implies$ 1.) Let $\varepsilon > 0$. Since $\lim_{x\to c^+} f(x) = L, \exists\delta_1 > -$ s.t. $0 < x - c < \delta_1 \implies |f(x) - L| < \varepsilon$. Since $\lim_{x\to c^-} f(x) = L, \exists\delta_2 > 0$ s.t. $\forall x \in A$ s.t. $-\delta_2 < x - c < 0 \implies |f(x) - L| < \varepsilon$. Take $\delta = \min\{\delta_1, \delta_2\}$. Then, if $0 < |x - c| < \delta$, then we have that either $0 < x - c < \delta \leq \delta_1$, or $-\delta_2 \leq -\delta < x - c < 0$. In either case, $|f(x) - L| < \varepsilon$, so $\lim_{x\to c} f(x) = L$ and 1. holds. ■

↪ **Theorem 3.4**

Let[56] $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and let c be a cluster point of the set $A \cap (c, \infty)$.Then, TFAE:

1. $\lim_{x\to c^+} f(x) = L$
2. For any sequence $(x_n) \in A$ s.t. $x_n > c \,\forall n$, and $\lim_{n\to\infty} x_n = c$, we have that $\lim_{n\to\infty} f(x_n) = L$.

Proof. ($\implies$)

($\impliedby$) ■

[56] Abbott, 4.3E1 (Theorem 4.3.2)

3.3 Limits and Infinity

3.3.1 Infinite Limits

↪ **Definition 3.5: Infinite Limits**

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, c a cluster point of A.

1. $\lim_{x\to c} f(x) = \infty$ if $\forall M \in \mathbb{R}, \exists\delta > 0$ s.t. $\forall x \in A$ s.t. $0 < |x - c| < \delta, f(x) \geq M$.
2. $\lim_{x\to c} f(x) = -\infty$ if $\forall M \in \mathbb{R}, \exists\delta > 0$ s.t. $\forall x \in A$ s.t. $0 < |x - c| < \delta, f(x) \leq M$.

⊛ **Example 3.10**

Let $A = (-\infty, 0) \cup (0, \infty)$ and let $f(x) = \frac{1}{x^2}$. Show that $\lim_{x\to 0} f(x) = \infty$.

Proof. Let $M \in \mathbb{R}$, and take $\delta = \frac{1}{\sqrt{|M|+1}}$. Then, $\forall x \in A$ s.t. $0 < |x| < \delta$, we have that

$$f(x) = \frac{1}{x^2} > \frac{1}{\delta^2} = |M| + 1 > M,$$

hence the limit holds. ■

⊛ **Example 3.11**

1. Give a sequential characterization of $\lim_{x\to c} f(x) = \infty$ and $-\infty$.

2. Give the definition of right/left hand limits going to infinity, $\lim_{x\to c^+} f(x) = \infty$ and $-\infty$, $\lim_{x\to c^-} f(x) = \infty$ and $-\infty$.

3. Let $A = (-\infty, 0) \cup (0, \infty)$, $f(x) = \frac{1}{x}$. Show that

$$\lim_{x\to 0^-} f(x) = -\infty,\ \lim_{x\to 0^+} f(x) = \infty.$$

↪ **Proposition 3.4: Order Properties of Infinite Limits**

Let $A \subseteq \mathbb{R}$, $f, g : A \to \mathbb{R}$, and suppose $f(x) \leq g(x)\,\forall\, x \in A$. Let c be a cluster point of A. Then,

1. $\lim_{x\to c} f(x) = \infty \implies \lim_{x\to c} g(x) = \infty$

2. $\lim_{x\to c} g(x) = -\infty \implies \lim_{x\to c} f(x) = -\infty$

Proof. (1.) Let $M \in \mathbb{R}$. Since $\lim_{x\to c} f(x) = \infty, \exists \delta > 0$ s.t. $\forall\, x \in A$ s.t. $0 < |x - c| < \delta$, $f(x) \geq M$. But then $g(x) \geq f(x)\,\forall\, x$, hence $g(x) \geq M \implies \lim_{x\to c} g(x) = \infty$. ■

3.3.2 Limits at Infinity

↪ **Definition 3.6: Limit at $\pm$ Infinity**

- Let $A \subseteq \mathbb{R}$. Suppose that for some $a \in \mathbb{R}$, $(a, \infty) \subseteq A$. Let $f : A \to \mathbb{R}$. We say that a real number L is the limit of f at ∞ if $\forall\, \varepsilon > 0\ \exists K > a$ s.t. $\forall\, x \geq K$, we have $|f(x) - L| < \varepsilon$.

- Let $A \subseteq \mathbb{R}$ and suppose that for some $a \in \mathbb{R}$, $(-\infty, a) \subseteq A$. Let $f : A \to \mathbb{R}$. We say that a real number L is the limit of f at $-\infty$ if $\forall\, \varepsilon > 0, \exists K < a$ s.t. $\forall\, x \leq K$, $|f(x) - L| < \varepsilon$.

⊛ **Example 3.12: $\frac{\sin x}{x}$ at infinity**

Let $A = (0, \infty)$ and $f(x) = \frac{\sin x}{x}$. Prove that $\lim_{x\to\infty} f(x) = 0$.

Proof. Let $\varepsilon > 0$. Take $K = \frac{2}{\varepsilon}$. Then, for $x \geq K$, $|f(x)| = \left|\frac{\sin x}{x}\right| \leq \frac{1}{x} \leq \frac{1}{K} = \frac{\varepsilon}{2} < \varepsilon$. ■

⊛ **Example 3.13**

"Sequentialize" limits at infinity.

⊛ **Example 3.14: Abbott, 4.4E9**

Prove that if $f : (a, \infty) \to \mathbb{R}$ is such that $\lim_{x\to\infty} xf(x) = L \in \mathbb{R}$ exists,

$$\lim_{x\to\infty} f(x) = 0.$$

Proof.

$$\begin{aligned}
\lim_{x\to\infty} xf(x) = L &\implies \forall \varepsilon > 0, \exists M \text{ s.t. } \forall x \geq M > 0, |xf(x) - L| < \varepsilon \\
&\implies L - \varepsilon < xf(x) < L + \varepsilon \\
&\implies \underbrace{\frac{L-\varepsilon}{x}}_{\to 0} < f(x) < \underbrace{\frac{L+\varepsilon}{x}}_{\to 0} \\
&\overset{\text{squeeze theorem}}{\implies} \lim_{x\to\infty} f(x) = 0
\end{aligned}$$

Noting that we take $M > 0$ wlog. ■

3.3.3 Infinite Limits at Infinity

↪ **Definition 3.7: Infinite Limits at Infinity**

3.4 Continuity

↪ **Definition 3.8: Continuity**

Let $A \subseteq \mathbb{R}$, $f : A \to \mathbb{R}$, and $c \in A$. We say f is *continuous* at c if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$ s.t. $|x - c| < \delta$, we have that $|f(x) - f(c)| < \varepsilon$. Quantified: f continuous at c if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in A)(|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon).$$

If f not continuous at some c, we say f *discontinuous* at c.

If $c \in A$ a cluster point of A, f is continuous at c iff $\lim_{x\to c} f(x) = f(c)$. If c not a cluster point, continuity at c still defined, while $\lim_{x \to c} f(x)$ not.

↪ **Theorem 3.5: Sequential Characterization of Continuity**

Let $f : A \to \mathbb{R}$, $c \in A$. TFAE:

1. f continuous at c
2. for any sequence $(x_n) \in A$ s.t. $\lim_{n\to\infty} x_n = c$, $\lim_{n\to\infty} f(x_n) = f(c)$

Remark 3.7.

Remark 3.8. *This theorem can be directly deduced from sequential characterization of functional limits.*

↪ **Proposition 3.5: Algebraic Operations**

Let $f, g : A \to \mathbb{R}, c \in A$. Suppose f, g continuous at c. Then:

1. $\forall\, k \in \mathbb{R}, kf$ continuous at c
2. $h = f + g$ continuous at c
3. $h = f \cdot g$ continuous at c
4. If $g(x) \neq 0\, \forall\, x \in A$, $h = \frac{f}{g}$ continuous at c.

Proof. (Of 3.) Let $(x_n) \in A$ s.t. $\lim_{n\to\infty} x_n = c$. Since f, g continuous at c, we have that $\lim_{n\to\infty} f(x_n) = f(c)$ and $\lim_{n\to\infty} g(x_n) = g(c)$. By algebraic properties of limits, then, $\lim_{n\to\infty} f(x_n) g(x_n) = f(c)g(c)$ and so $\forall\, (x_n) \in A$ s.t. $\lim_{n\to\infty} x_n = c$, we have that $\lim_{n\to\infty} h(x_n) = h(c)$ where $h = f \cdot g$ and thus h continuous at c. ■

↪ **Theorem 3.6: Composition of Functions and Continuity**

Let $f : A \to \mathbb{R}, g : B \to \mathbb{R}$ be two functions such that

$$f(A) = \{f(x) : x \in A\} \subseteq B$$

so that the composite function $h(x) = g \circ f(x) = g(f(x))$ is well defined on A. Suppose $c \in A$ such that f continuous at c and g continuous at $f(c)$. Then, h also continuous at c.

Proof. (Using sequential characterization) Let $(x_n) \in A$ s.t. $\lim_{n\to\infty} x_n = c$. f continuous at c, so $\lim_{n\to\infty} f(x_n) = f(c)$. Let $(f(x_n))_{n\geq 1}$ is a sequence in B such that $\lim_{n\to\infty} f(x_n) = f(c)$ and so g is continuous at $f(c)$. Then, $\lim_{n\to\infty} g(f(x_n)) = g(f(c))$ so $\forall\,(x_n) \in A$ s.t. $\lim_{n\to\infty} x_n = c$. We thus have that $\lim_{n\to\infty} h(x_n) = h(c)$ where $h = g \circ f$, hence h is continuous at c.

$(\varepsilon - \delta)$ Fix $\varepsilon > 0$. Since g is continuous at $f(c)$, $\exists \delta' > 0$ s.t. $\forall\, y \in B$ s.t. $|y - f(c)| < \delta$, $|g(y) - g(f(c))| < \varepsilon$.

Since f continuous at c, $\exists \delta' > 0$ s.t. $\forall\, x \in A$ s.t. $|x - c| < \delta'$, $|f(x) - f(c)| < \delta'$. Then, for such x, $|g(f(x)) - g(f(c))| < \varepsilon$ and the proof is complete, taking $h = g \circ f$. ■

⊛ **Example 3.15**

$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$.

Proof. ■

⊛ **Example 3.16:** $f(x) = \sin x$

Show that $f(x) = \sin x$ continuous on any $c \in \mathbb{R}$.

Proof. Fix $\varepsilon > 0$. Take $\delta = \varepsilon$, and take x s.t. $|x - c| < \delta$. Then,

$$\begin{aligned} |\sin x - \sin c| &= \left|2 \sin\left(\frac{x-c}{2}\right) \cos\left(\frac{x+c}{2}\right)\right| \\ &= 2\left|\sin\left(\frac{x-c}{2}\right)\right|\left|\cos\left(\frac{x+c}{2}\right)\right| \\ &\leq 2\left|\frac{x-c}{2}\right| = |x - c| < \delta = \varepsilon. \end{aligned}$$

■

⊛ **Example 3.17: Dirichlet Function**

Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Show f discontinuous $\forall\, c \in \mathbb{R}$.

Proof. Fix $c \in \mathbb{R}$ and let $n \in \mathbb{N}$. Consider the interval $(c - \frac{1}{n}, c + \frac{1}{n})$. By density of the rationals, there must exist some $x_n \in \mathbb{Q}$ s.t. $x_n \in (c - \frac{1}{n}, c + \frac{1}{n})$, and similarly, by density of the irrationals, there must exists some $y_n \in \mathbb{J}$, $y_n \in (c - \frac{1}{n}, c + \frac{1}{n})$.

We have, then,

$$|x_n - c| < \frac{1}{n} \text{ and } |y_n - c| < \frac{1}{n},$$

and moreover, $f(x_n) = 1$ and $f(y_n) = 0 \,\forall\, n$. We also have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = c$, and so f cannot be continuous. ■

⊛ **Example 3.18: Thomae's Function**

Let $f : \mathbb{R}^+ \to \mathbb{R}, x \mapsto \begin{cases} 0 & x \in \mathbb{J} \\ \frac{1}{n} & x = \frac{m}{n} \in \mathbb{Q}, \gcd(m,n) = 1 \end{cases}$.

Show f discontinuous for any $a \in \mathbb{Q}$ and continuous for any $a \in \mathbb{J}$.

Proof. Let $a > 0$ be rational. Then, $f(a) > 0$, by construction of the function. Let $(x_n) \in \mathbb{J}$ s.t. $(x_n) \to a$. Then, $\lim_{n\to\infty} f(x_n) = 0$, despite $f(a) > 0$, hence f is not continuous at a. ■

3.4.1 Extensions By Continuity

↪ **Definition 3.9: Extension by Continuity**

Let $f : A \to \mathbb{R}$, c a cluster point of A s.t. $c \notin A$. Since $c \notin A$, we cannot say whether f continuous or not at a, but we can *extend* f to $A \cup \{c\}$ by setting

$$F(x) := \begin{cases} f(x) & x \in A \\ L & x = c \end{cases}.$$

Remark 3.9. *Since c a cluster point of $A \cup \{c\}$, we have that F continuous at c iff $\lim_{x\to c} F(x) = L \iff \lim_{x\to c} f(x) = L$. Hence, if $f : A \to \mathbb{R}$, c a cluster point of A, $c \notin A$, and $\lim_{x\to c} = L$, F is continuous at c. If $\lim_{x\to c} f(x)$ DNE, f cannot be extended.*

⊛ **Example 3.19**

$f(x) = x \sin \frac{1}{x}$, defined on $A = (-\infty, 0) \cup (0, \infty)$. 0 a cluster point of A. Note that $\lim_{x\to 0} f(x) = 0$, since $|f(x)| \leq |x|$, so if we extend f to 0 by setting $f(0) = 0$, then the extended function is continuous at 0.

3.5 Continuity on Bounded & Closed Interval

↪ **Definition 3.10: Bounded Function**

Let A be a set. A function $f : A \to \mathbb{R}$ is called *bounded* if $\exists M > 0$ s.t. $|f(x)| \leq M \,\forall\, x \in A$.

↪ **Theorem 3.7: Closed Domain & Continuous Implies Bounded**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, f is bounded.

Proof. We proceed by contradiction. Suppose there exists a continuous function $f : [a, b] \to \mathbb{R}$ that is not bound. Then, for any $n \in \mathbb{N}$, it is *not* true that $|f(x)| \leq n \,\forall\, x \in [a, b]$ (otherwise, this n would be a bound).

So, for any n, $\exists x_n \in [a, b]$ s.t. $|f(x_n)| > n$. Then, (x_n) is a sequence in $[a, b]$ and by the Bolzano-Weirestrass Theorem, this sequence has a subsequence (x_{n_k}) that converges to some $x \in [a, b]$, that is,

$$\lim_{k \to \infty} x_{n_k} = x.$$

So, by the sequential characterization of continuity, we have then that

$$\lim_{k \to \infty} f(x_{n_k}) = f(x).$$

Hence, $(f(x_{n_k}))$ is a converging sequence of real numbers. But by the construction of x_n, we have

$$|f(x_{n_k})| > n_k \geq k,$$

so $(f(x_{n_k}))$ is a converging sequence of real numbers that is not bounded, which contradicts the fact that any converging sequence is bounded. ■

⊛ **Example 3.20: "Not Closed" Domain**

Consider the function $f : (0, 1] \to \mathbb{R}, x \mapsto \frac{1}{x}$. This function is continuous, and the interval $(0, 1]$ is bounded but *not* closed. Hence, the function is *not* bounded on this interval; for any $M > 0$, if we take $x \in (0, 1]$ such that $0 < x < \frac{1}{M+1}$, we have that $f(x) = \frac{1}{x} > M + 1 > M$.

↪ **Definition 3.11: Absolute Max/Min**

Let $f : A \to \mathbb{R}$. We say that f has *absolute maximum* at $\overline{x} \in A$ if $f(\overline{x}) \geq f(x) \,\forall\, x \in A$. f has *absolute minimum* at $\underline{x} \in A$ if $f(\underline{x}) \leq f(x) \,\forall\, x \in A$.

↪ **Theorem 3.8**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, f has an absolute maximum and absolute minimum on $[a, b]$.

Proof. (of absolute maximum) Consider the set

[56] Absolute minimum case follows by taking $-f$.

$$f([a, b]) = \{f(x) : x \in [a, b]\}.$$

By theorem 3.7, f is bounded on $[a, b]$, so there exists some $M > 0$ such that

$$f([a, b]) \subseteq [-M, M].$$

So, the set $f([a, b])$ is bounded, and by Axiom Of Completeness, $s = \sup(f([a, b]))$ exists. Hence, $s \geq f(x) \, \forall \, x \in [a, b]$. We aim to show then that $\exists \overline{x} \in [a, b]$ s.t. $s = f(\overline{x})$.

Let $n \in \mathbb{N}$. Since $s - \frac{1}{n}$ is not an upper bound of $f([a, b])$,

$$\exists x_n \in [a, b] \text{ s.t. } s - \frac{1}{n} < f(x_n) \leq s. \quad \circledast$$

By Bolzano-Weirestrass Theorem, (x_n) has a converging subsequence (x_{n_k}). Let $\overline{x} = \lim_{k \to \infty} x_{n_k}$. By the sequential characterization of continuity, then, we have that $f(\overline{x}) = \lim_{k \to \infty} f(x_{n_k})$. By $\circledast$, we have

$$s - \frac{1}{n_k} < f(x_{n_k}) \leq s.$$

Moreover, we have that $n_k \geq k \implies \frac{1}{n_k} \leq \frac{1}{k} \implies -\frac{1}{n_k} \geq -\frac{1}{k}$. Hence,

$$s - \frac{1}{k} < f(x_{n_k}) \leq s,$$

and so by the squeeze theorem, $\lim_{k \to \infty} f(x_{n_k}) = s = f(\overline{x})$, and the proof is complete. ■

↪ **Theorem 3.9: Location of the Roots**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that

$$f(a) < 0 < f(b).$$

Then, $\exists c$ s.t. $a < c < b$ and $f(c) = 0$.

Proof. Let $S = \{x \in [a, b] : f(x) \leq 0\}$. $S \neq \varnothing$ since $a \in S$. S also bounded (it is a subset of a bounded interval). Let $c = \sup S$ (exists by AC). We claim this c is the point as defined in the theorem; we aim to show that $f(c) = 0$.

Let $\varepsilon = \min\{\frac{|f(a)|}{2}, \frac{f(b)}{2}\} > 0$. Since f continuous at a, $\exists \delta' > 0$ s.t. $\forall x \in [a,b]$ s.t. $|x-a| < \delta'$, we have $|f(x) - f(a)| < \varepsilon$. Since f is continuous at b, $\exists \delta'' > 0$ s.t. $\forall x \in [a,b]$ s.t. $|x - b| < \delta''$, we have $|f(x) - f(b)| < \varepsilon$. Let $\delta = \min\{\delta', \delta'', \frac{b-a}{2}\}$. Then, $\forall x \in [a, a+\delta)$, we have that

$$f(x) - f(a) < \varepsilon \leq \frac{|f(a)|}{2} \implies f(x) < \frac{|f(a)|}{2} + f(a) = \frac{f(a)}{2} < 0.$$

So, $\forall x \in [a, a+\delta)$, $f(x) < 0$ and thus $[a, a+\delta) \subseteq S$. Hence, c, being the supremum of S, must have that $c \geq a + \delta > 0$.

Since $\delta \leq \delta''$, we have that $\forall x \in (b - \delta, b]$,

$$f(x) - f(b) > -\varepsilon \geq -\frac{f(b)}{2} \implies f(x) > f(b) - \frac{f(b)}{2} = \frac{f(b)}{2} > 0.$$

So, $\forall x \in (b-\delta, b]$, we have that $f(x) > 0$. So, if we take $[b - \frac{\delta}{2}, b]$, then for every $x \in$ this interval $f(x) > 0$ and so $S \subseteq [a, b - \frac{\delta}{2})$, and thus $c = \sup S \leq b - \frac{\delta}{2}$. Hence, $\exists \delta$ such that $a + \delta \leq c \leq b - \frac{\delta}{2}$. So, c satisfies $a < c < b$.

We now show that $f(c) \leq 0$ and $f(c) \geq 0$ and so $f(c) = 0$.

($f(c) \leq 0$) Let $n \in \mathbb{N}$ and consider $c - \frac{1}{n}$; this is not an upper bound of S, so $\exists (x_n) \in S$ s.t. $c - \frac{1}{n} < x_n \leq c$. This gives us a sequence such that $\lim_{n\to\infty}(x_n) = c$. Since f is continuous, by the sequential characterization of continuity, we have that $\lim_{n\to\infty} f(x_n) = f(c)$. Moreover, $x_n \in S$ and thus $f(x_n) \leq 0$ (by construction of S), hence $f(c) \leq 0$.

($f(c) \geq 0$) Since $c < b$, we can find $(x_n) \in [a,b]$ s.t. $x_n > c$, $\lim_{n\to\infty} x_n = c$ ($x_n = c + \frac{1}{n}$, for instance). We must have that $f(x_n) > 0$; otherwise, $f(x_n) \leq 0 \implies x_n \in S$, and since we have $x_n > c$ (by construction), this would contradict the fact that c an upper bound for S. So, we have that $\lim_{n\to\infty} x_n = c \implies \lim_{n\to\infty} f(x_n) = f(c) \geq 0$, that is, $f(c) \geq 0$.

Thus, having show both $f(c) \leq 0$ and $f(c) \geq 0$, we conclude that $\exists c \in [a,b]$ s.t. $a < c < b$, where $f(c) = 0$, and the proof is complete. ■

3.6 Intervals in $\mathbb{R}$

↪ **Definition 3.12: Types of Intervals in $\mathbb{R}$**

(Bounded Intervals)

- $[a,b] = \{x : a \leq x \leq b\} \subseteq \mathbb{R}$

- $(a,b) = \{x : a < x < b\} \subseteq \mathbb{R}$
- $[a,b) = \{x : a \leq x < b\} \subseteq \mathbb{R}$
- $(a,b] = \{x : a < x \leq b\} \subseteq \mathbb{R}$

(Unbounded Intervals)

- $[a,\infty) = \{x : x \geq a\}$
- $(a,\infty) = \{x : x > a\}$
- $(-\infty,a] = \{x : x \leq a\}$
- $(-\infty,a) = \{x : x < a\}$
- $\mathbb{R} = (-\infty,\infty)$

Remark 3.10. *If you take any interval and any two points $x < y$ in the interval, then $[x,y]$ is completely contained within the given interval.*

↪ **Theorem 3.10**

Let $S \subseteq \mathbb{R}$ that contains more than two points. Suppose S has the property that $\forall\, x, y \in S$ s.t. $x < y$, $[x,y] \subseteq S$. Then, S is an interval.

Proof. Suppose S bounded. Then, $a = \inf S, b = \sup S$ exist. Then, for any $x \in S$, $a \leq x \leq b$, so $S \subseteq [a,b]$. Let now $a < z < b$. $z < b \implies z$ not an upper bound of S so $\exists y \in S$ s.t. $z < y$. $z > a \implies z$ not a lower bound of S so $\exists x \in S$ s.t. $x < z$. Then, $x < z < y, x, y \in S$, so $[x,y] \subseteq S \implies z \in S$. So, $(a,b) \subseteq S \subseteq [a,b]$ and thus S must be a bounded interval (one of those types defined above). ■

↪ **Theorem 3.11: Bolzano's Intermediate Value Theorem**

Let I be an interval and $f : I \to \mathbb{R}$ a continuous function. Let $a, b \in I$ and suppose $f(a) < f(b)$. Then, for any k s.t. $f(a) < k < f(b)$, $\exists c$ between a and b s.t. $f(c) = k$.

Proof.

- (Case 1 : $a < b$) Consider $h(x) = f(x) - k$ on the closed and bounded interval $[a,b]$. Note that $h(a) = f(a) - k < 0$, and $h(b) = f(b) - k > 0$. By Location of the Roots, there exists a $a < c < b$ s.t. $h(c) = 0 = f(c) - k \implies f(c) = k$, as desired.
- (Case 2: $a > b$) Consider $h(x) = k - f(x)$ on the closed and bounded interval $[b,a]$. The remainder of the proof follows identically to (Case 1).

■

↪ **Theorem 3.12**

Let $I = [a, b]$ and $f : I \to \mathbb{R}$ a continuous function. Let k be s.t. $\inf f(I) \leq k \leq \sup f(I)$. Then, $\exists c \in I$ s.t. $f(c) = k$.

Proof. Recall that $m = \inf f(I)$ is the absolute minimum of f on I and $M = \sup f(I)$ is the absolute maximum of f on I. Moreover, $\exists \overline{x}, \underline{x} \in I$ s.t. $f(\overline{x}) = M, f(\underline{x}) = m$. Hence, we have that our k satisfies

$$f(\underline{x}) \leq k \leq f(\overline{x}).$$

If $k = f(\underline{x})$, take $c = \underline{x}$. If $k = f(\overline{x})$, take $c = \overline{x}$. Otherwise, the inequality is strict, and we have $f(\underline{x}) < k < f(\overline{x})$. By Bolzano's Intermediate Value Theorem, we have that $\exists c$ between $\underline{x}$ and $\overline{x}$ s.t. $f(c) = k$. Moreover, $c \in [a, b]$. ■

↪ **Theorem 3.13**

Let $I = [a, b]$ and $f : I \to \mathbb{R}$ a continuous function. Then, $f(I)$ is also a bounded and closed interval.

Proof. Let $m = \inf f(I)$, $M = \sup f(I)$. Then, for any $x \in [a, b]$, $m \leq f(x) \leq M$, hence, $f(I) \subseteq [m, M]$. OTOH, by theorem 3.12, for any $m \leq k \leq M$, $\exists c \in [a, b]$ s.t. $f(c) = k$, hence, $[m, M] \subseteq f(I)$, and thus $f(I) = [m, M]$ and the proof is complete. Moreover, $f(I)$ is precisely $[\inf f(I), \sup f(I)]$. ■

↪ **Theorem 3.14**

Let I be an interval in $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be a continuous function. Then, $f(I)$ is also an interval.

Proof. We assume f is not just a constant function. We aim to show that if $\alpha, \beta \in f(I), \alpha < \beta$, then $[\alpha, \beta] \subseteq f(I)$, that is, $f(I)$ an interval.

Let $a, b \in I$ be such that $f(a) = \alpha, f(b) = \beta$. We have that $f(a) < f(b)$, so for any k s.t. $\alpha \leq k \leq \beta$, by Bolzano's Intermediate Value Theorem, $\exists c \in I$ s.t. $f(c) = k$. Hence, $[\alpha, \beta] \subseteq f(I)$, and the proof is complete. ■

Remark 3.11. *This argument does not specify the actual "shape" of the intervals $f(I)$ look like.*

- *If $I = \mathbb{R}$, can $f(I)$ be bounded and closed? Yes; take $f(x) = \sin x$; then, $f(\mathbb{R}) = [-1, 1]$*
- *If $I = (a, b)$, can $f(I) = \mathbb{R}$? Yes; take $I = (-\frac{\pi}{2}, \frac{\pi}{2})$, $f(x) = \tan x$.*

3.7 Uniform Continuity

Remark 3.12. *Recall that in the definition of continuity, the "choice" of δ depended both on c (the point in the domain) and ε. Uniform continuity defines a manner in which δ can be chosen* without *relying on c; if this is the case for a function $f : A \to \mathbb{R}$, we say that f is uniformly continuous on A.*

↪ **Definition 3.13: Uniform Continuity**

Let $f : A \to \mathbb{R}$. We say f is *uniformly continuous* on A if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall c \in A)(\forall x \in A)(|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon).$$

Remark 3.13. *The difference, quantifiers-wise, is the position of the $(\forall c \in A)$; since here the "choice" of c comes after the choice of δ, δ is independent, in contrast with "local" continuity.*

⊛ **Example 3.21**

Let $f(x) = x$. Then, f is uniformly continuous on $\mathbb{R}$.

Proof. Let $\varepsilon > 0, \delta = \varepsilon$. Then, $\forall c \in \mathbb{R}$, if x s.t. $|x - c| < \delta$, then we have $|f(x) - f(c)| = |x - c| < \varepsilon$. ■

⊛ **Example 3.22**

Let $f(x) = x^2$. Then, f is not uniformly continuous on $\mathbb{R}$.

Proof. We proceed by contradiction. Suppose f uniformly continuous. Take $\varepsilon = 1$, then, $\exists \delta > 0$ s.t. $\forall x, c \in A$ s.t. $|x - c| < \delta$, $|f(x) - f(c)| = |x^2 - c^2| < 1$. Take $x = \frac{1}{\delta} + \delta, c = \frac{1}{\delta} + \frac{\delta}{2}$. Then, we have

$$|x - c| = \frac{\delta}{2} < \delta,$$

but

$$\begin{aligned} |x^2 - c^2| &= \left|(\frac{1}{\delta} + \delta)^2 - (\frac{1}{\delta} + \frac{\delta}{2})^2\right| \\ &= \cdots = \left|1 + \frac{3\delta^2}{4}\right| > 1, \end{aligned}$$

a contradiction. ∎

⊛ **Example 3.23**

Let $f(x) = \sqrt{x}$. Then, f is uniformly continuous on $[0, \infty)$.

Proof. Let $\varepsilon > 0$. Take $\delta = \frac{\varepsilon^2}{2}$. Let $x, c \geq 0$, and suppose $|x - c| < \delta$. We consider two cases:

- (Case 1) $x, c \in [0, \frac{\varepsilon^2}{4})$. Then,

$$\begin{aligned} \left|\sqrt{x} - \sqrt{c}\right| &\leq \sqrt{x} + \sqrt{c} \\ &< \sqrt{\frac{\varepsilon^2}{4}} + \sqrt{\frac{\varepsilon^2}{4}} = 2 \cdot \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

- (Case 2) Either x or $c \geq \frac{\varepsilon^2}{4}$. then

$$\begin{aligned} \left|\sqrt{x} - \sqrt{c}\right| = \left|(\sqrt{x} - \sqrt{c})\frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}}\right| &= \frac{|x - c|}{\sqrt{x} + \sqrt{c}} \\ &< \frac{\delta}{\frac{\varepsilon}{2}} = \frac{\frac{\varepsilon^2}{2}}{\frac{\varepsilon}{2}} = \varepsilon. \end{aligned}$$

∎

⊛ **Example 3.24**

Let $f(x) = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous on $(0, 1]$.

Proof. Suppose that f is indeed uniformly continuous. Take $\varepsilon = \frac{1}{2}$. Then, $\exists \delta > 0$ s.t. $\forall x, c \in (0, 1]$ s.t. $|x - c| < \delta, |f(x) - f(c)| = |\sin x - \sin c| < \frac{1}{2}$. Take $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \delta$. Take $x = \frac{1}{n\pi}, c = \frac{1}{(2n+1)\frac{\pi}{2}} = \frac{1}{n\pi + \frac{\pi}{2}}$. Then,

$$|x - c| = \left|\frac{1}{n\pi} - \frac{1}{n\pi + \frac{\pi}{2}}\right| = \frac{\frac{\pi}{2}}{n\pi(n\pi + \frac{\pi}{2})} < \delta$$

Then, we have

$$\begin{aligned} f(x) - f(c) &= \left|\sin\frac{1}{\frac{1}{n\pi}} - \sin\frac{1}{\frac{1}{(2n+1)\frac{\pi}{2}}}\right| \\ &= |(-1)^n| = 1 > \frac{1}{2}, \end{aligned}$$

a contradiction. ∎

3.8 Sequential Characterization of Non-Uniform Continuity

↪ **Theorem 3.15**

Let $f : A \to \mathbb{R}$ be a continuous function. TFAE:

1. f is *not* uniformly continuous on A;
2. $\exists \varepsilon_0 > 0$ and two sequences $(x_n), (y_n) \in A$ s.t. $\lim_{n\to\infty}(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0 \, \forall \, n$.

Proof. (1. $\implies$ 2.) For f to be *not* uniformly continuous, then it is *not* true that $\forall \, \varepsilon > 0 \exists \delta > 0$ s.t. $\forall \, x, y \in A$, if $|x - y| < \delta, |f(x) - f(y)| < \varepsilon$. That is, $\exists \varepsilon_0 > 0$ s.t. $\forall \, \delta > 0$, one can find $x, y \in A$ s.t. $|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon_0$.

Take this ε_0 and let $\delta = \frac{1}{n}$. Then, $\exists x_n, y_n \in A$ s.t. $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0$. This defines sequences $(x_n), (y_n) \in A$ s.t. $\lim_{n\to\infty}(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0 \, \forall \, n$, hence 2. holds.

(2. $\implies$ 1.) We argue by contradiction. Suppose there $\exists f$ continuous, $f : [a, b] \to \mathbb{R}$ s.t. 2. holds but 1. does not; that is, f uniformly continuous and $\exists \varepsilon_0 > 0$ and $(x_n), (y_n) \in A$ s.t. $\lim(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0 \, \forall \, n$.

Take this ε_0 in the definition of uniform continuity; then, if f uniformly continuous, $\exists \delta > 0$ s.t. $\forall \, x, y \in A$ s.t. $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon_0$. Consider our $(x_n), (y_n)$. Since $\lim_{n\to\infty}(x_n - y_n) = 0, \exists N$ s.t. $\forall \, n \geq N, |x_n - y_n| < \delta$. But then, this implies that $\forall \, n \geq N$, we have that $|f(x_n) - f(y_n)| < \varepsilon_0$. But this contradicts our original assumption in 2., and hence 1. must hold and the proof is complete. ■

⊛ **Example 3.25:** $f(x) = x^2$

Show that $f(x) = x^2$ *not* uniformly continuous on $\mathbb{R}$.

Proof. Take $x_n = n + \frac{1}{n}, y_n = n$. Then, $x_n - y_n = \frac{1}{n}$ hence $\lim_{n\to\infty}(x_n - y_n) = 0$. OTOH,

$$f(x_n) - f(y_n) = n^2 + 2 + \frac{1}{n^2} - n^2 = 2 + \frac{1}{n^2} \geq 2,$$

hence, by the sequential characterization, taking $\varepsilon_0 = 2$, f is not uniformly continuous on $\mathbb{R}$. ■

⊛ **Example 3.26:** $f(x) = \sin \frac{1}{x}$

Show that $f(x) = \sin \frac{1}{x}$ *not* uniformly continuous on $(0, 1]$

Proof. Let $x_n = \frac{1}{n\pi + \frac{\pi}{2}}, y_n = \frac{1}{n\pi}$. Both of these converge to 0, hence their differences do as well. OTOH, $|f(x_n) - f(y_n)| = |-1 - 0| = 1 \geq 1$, hence, f is not uniformly continuous with $\varepsilon_0 = 1$.s ■

↪ **Theorem 3.16**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, f is uniformly continuous on $[a, b]$.

Proof. We proceed by contradiction. Suppose $\exists f : [a, b] \to \mathbb{R}$ that is continuous but not uniformly continuous on $[a, b]$. Then, by the sequential characterization, $\exists \varepsilon_0 > 0$ and $x_n, y_n \in [a, b]$ s.t. $\lim_{n\to\infty}(x_n - y_n) = 0$ and $|f(x_n) - f(y_n)| \geq \varepsilon_0 \, \forall \, n$.

Since $[a, b]$ bounded, by Bolzano-Weirestrass Theorem, the sequence (x_n) has a convergent subsequence (x_{n_k}) that converges to $z \in [a, b]$ (since $[a, b]$ closed). We can write

$$\begin{aligned} |y_{n_k} - z| &= |y_{n_k} - x_{n_k} + x_{n_k} + z| \\ &\leq \underbrace{|y_{n_k} - x_{n_k}|}_{\to 0} + \underbrace{|x_{n_k} - z|}_{\to 0}. \end{aligned}$$

Hence, by the The Squeeze Theorem, $|y_{n_k} - z|$ converges to 0 so (y_{n_k}) also converges to z, that is

$$\lim_{k\to\infty} x_{n_k} = \lim_{k\to\infty} y_{n_k} = z.$$

Since f continuous on $[a, b]$, we have that $\lim_{k\to\infty} f(x_{n_k}) = f(z)$ and $\lim_{k\to\infty} f(y_{n_k}) = f(z)$, and so

$$\lim_{k\to\infty} (f(x_{n_k}) - f(y_{n_k})) = z - z = 0,$$

By definition, then, $\exists K$ s.t. $\forall \, k \geq K$, $|f(x_{n_k}) - f(y_{n_k})| < \varepsilon_0$. This is a contradiction, hence, f uniformly continuous and the proof is complete. ■

↪ **Theorem 3.17: Preservation of Cauchy Criterion by Uniformly Continuous Functions**

Let $f : A \to \mathbb{R}$ be a uniformly continuous function. Let $(x_n) \in A$, and assume x_n Cauchy. Then, $(f(x_n))$ is also a Cauchy sequence.

Proof. Let $\varepsilon > 0$. Since f uniformly continuous on A, there is $\delta > 0$ s.t. $\forall \, x, y \in A, |x - y| <$

$\delta \implies |f(x) - f(y)| < \varepsilon$. Since (x_n) Cauchy, $\exists N$ s.t. $\forall n, m \geq N$, $|x_n - x_m| < \delta$. But then, $\forall n, m \geq N$, we have $|f(x_n) - f(y_n)| < \varepsilon$, and hence $(f(x_n))$ is Cauchy. ■

↪ Theorem 3.18: Continuous Extension Theorem

Let (a, b) be an bounded, *open* interval and $f : (a, b) \to \mathbb{R}$ a continuous function. TFAE:

1. f is uniformly continuous on (a, b);
2. f can be at the end points a, b such that it is continuous on the closed interval $[a, b]$.

Proof. (2. $\implies$ 1.) If f can be extended to a, b so that it is continuous on $[a, b]$, then it is also uniformly continuous by theorem 3.16. Then, f is also uniformly continuous on any subset $[a, b]$, in particular, on $(a, b) \subseteq [a, b]$.

(1. $\implies$ 2.) Let $(x_n) \in (a, b)$ that converges to a. (x_n) Cauchy, and by theorem 3.17, $(f(x_n))$ also Cauchy, hence $L = \lim_{n\to\infty} f(x_n)$ exists; define ("extend") $f(a) = L$. It remains to show that f continuous with this extension.

Let (u_n) be an arbitrary sequence in (a, b) such that $\lim_{n\to\infty} u_n = a$. Let $\varepsilon > 0$. Since f is uniformly continuous on (a, b), then $\exists \delta > 0$ s.t. $\forall x, y \in (a, b)$, $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$. Now, we have that $\lim_{n\to\infty} u_n = a$, $\lim_{n\to\infty} x_n = a$, so $\lim_{n\to\infty}(u_n - x_n) = 0$. Hence, $\exists N_1$ s.t. $\forall n \geq N_1, |u_n - x_n| < \delta$. This implies, then, that $\forall n \geq N_1$, we have $|f(u_n) - f(x_n)| < \frac{\varepsilon}{2}$.

We have, by our extension, that $\lim_{n\to\infty} f(x_n) = L$, hence, $\exists N_2$ s.t. $\forall n \geq N_2, |f(x_n) - L| < \frac{\varepsilon}{2}$. Let, now, $N = \max\{N_1, N_2\}$. Then, $\forall n \geq N$,

$$\begin{aligned} |f(u_n) - L| &= |f(u_n) - f(x_n) + f(x_n) - L| \\ &\leq |f(u_n) - f(x_n)| + |f(x_n) - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

that is, $\forall n \geq N, |f(u_n) - L| < \varepsilon$. Hence, for any arbitrary $(u_n) \in (a, b)$ such that $(u_n) \to a$, $\lim n \to \infty u_n = L$, hence, as we have set $f(a) = L$, by sequential characterization of continuity, f is continuous at a.

The proof for b, the RHS endpoint, follows identically. ■

3.9 Monotone and Inverse Functions

↪ **Definition 3.14: Increasing/Decreasing Function**

Let $f : A \to \mathbb{R}$. We say f is:

- *increasing* on A if $\forall\, x, y \in A, x \leq y \implies f(x) \leq f(y)$;
- *strictly increasing* on A if $\forall\, x, y \in A, x < y \implies f(x) < f(y)$;
- *decreasing* on A if $\forall\, x, y \in A, x \leq y \implies f(x) \geq f(x)$;
- *strictly decreasing* on A if $\forall\, x, y \in A, x < y \implies f(x) > f(y)$.

A function that is either increasing or decreasing is called *monotone*. If this increasing or decreasing is strict, the function is called *strictly monotone*.

↪ **Proposition 3.6**

$f : A \to \mathbb{R}$ increasing on $A \iff g = -f$ decreasing on A.

Remark 3.14. *Analogous statements hold for decreasing/strictly increasing/decreasing etc. The remaining theorems/propositions will be discussed with respect to increasing functions; the same concepts apply (with reversed inequalities, etc) to decreasing functions.*

↪ **Theorem 3.19**

Let $I \subseteq \mathbb{R}$, $f : I \to \mathbb{R}$ be increasing. Let $c \in I$, where c not an endpoint of I. Then:

1. $\lim_{c \to c^-} f(x) = \sup\{f(x) : x \in I, x < c\}$
2. $\lim_{c \to c^+} f(x) = \inf\{f(x) : x \in I, x > c\}$

Proof. We prove for 2.; 1. follows identically. Let $A := \{f(x) : x \in I, x > c\}$. Note that $A \neq \varnothing$, since c not an endpoint of I by construction and hence $\exists x \in I$ s.t. $x > c$.

Since f increasing, we have that $x > c \implies f(x) \geq f(c)$ hence A bounded below by $f(c)$, and thus $L := \inf A$ exists. Let $\varepsilon > 0$; since $L + \varepsilon$ not a lower bound for A, there exists some $x_\varepsilon \in I$ s.t. $L + \varepsilon > f(x_\varepsilon) \geq L$. Take $\delta = x_\varepsilon - c$. Since f increasing, we have that

$$c < x < c + \delta = x_\varepsilon \implies |f(x) - L| = f(x) - L \leq f(x\varepsilon) - L < \varepsilon.$$

But this is just the definition of the right hand limit, hence

$$\lim_{x \to c^+} f(x) = L.$$

■

↪ Corollary 3.1

Let $I \subseteq \mathbb{R}$ and $f : I \to \mathbb{R}$ be increasing on I. Take $c \in I$ such that c not an endpoint of I. TFAE:

1. f continuous at c
2. $\lim_{x \to c^-} f(x) = f(c) = \lim_{x \to c^+} f(x)$
3. $\sup\{f(x) : x \in I, x < c\} = f(c) = \inf\{f(x) : x \in I, x > c\}$.

Proof. Note first that 1. $\iff$ 2. does not relate to f increasing; rather, it follows from the left-hand limit equals right-hand limit iff limit holds; this holds if f continuous at c.

2. $\iff$ 3. follows from theorem 3.19. ■

↪ Definition 3.15: Jump

Let $f : I \to \mathbb{R}$ be increasing on I. If $c \in I$ not an endpoint of I, the *jump* of f at c is defined

$$j_f(c) = \lim_{x \to c^+} f(x) - \lim_{x \to c^-} f(x).$$

If c the left endpoint of I, then we define

$$j_f(c) = \lim_{x \to c^+} f(x) - f(c),$$

and if c the right endpoint of I,

$$j_f(c) = f(c) - \lim_{x \to c^-} f(x).$$

Remark 3.15. *It follows naturally that f continuous at $c \in I \iff j_f(c) = 0$.*

↪ Theorem 3.20

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be increasing. Then the set $D \subseteq I$ of points at which f is *discontinuous* is either finite or countable.

Proof. We will prove this result in the case that $I = [a, b]$, and deduce the remaining cases.

Note first that $j_f(c) \geq 0 \,\forall\, c \in I$. Consider some n points in I,

$$a \leq x_1 < x_2 < \cdots x_n \leq b.$$

We claim that the following inequality holds:

$$j_f(x_1) + j_f(x_2) + \cdots j_f(x_n) \leq f(b) - f(a).$$

Indeed, we have that

$$\begin{aligned} j_f(x_1) + \cdots + j_f(x_n) &= \lim_{x \to x_1^+} f(x) - \lim_{x \to x_1^-} f(x) + \cdots \lim_{x \to x_n^+} f(x) - \lim_{x \to x_n^-} f(x) \\ &= \lim_{x \to x_n^+} f(x) - \lim_{x \to x_1^-} f(x) + \sum_{k=1}^{n-1} \underbrace{\left(\lim_{x \to x_k^+} f(x) - \lim_{x \to x_{k+1}^-} f(x) \right)}_{\leq 0} \\ &\leq \lim_{x \to x_k^+} f(x) - \lim_{x \to x_1^-} f(x) \\ &\leq f(b) - f(a) \quad \circledast \end{aligned}$$

From this, we have that for any $k \in \mathbb{N}$, there are at most k points in I such that $j_f(x) \geq \frac{f(b)-f(a)}{k}$; suppose there were $k+1$ points; then,

$$f(b) - f(a) \geq j_f(x_1) + \cdots + j_f(x_{k+1}) \geq \frac{k+1}{k}(f(b - f(a))) > f(b) - f(a) \bot.$$

Let $D := \{x \in I : f \text{ discontinuous at } x\} = \{x \in I : j_f(x) > 0\} = \bigcup_{k=1}^{\infty} \{x \in I : j_f(x) \geq \frac{f(b)-f(a)}{k}\}$. This is a countable union of finite sets, hence D itself is finite or countable given $I = [a, b]$.

We now prove for general I. Any interval I can be written as

$$I = \bigcup_{k=1}^{\infty} [a_k, b_k],$$

for some sequences a_n, b_n, that is, as a countable union of bounded and closed intervals $\ominus$. Hence, we can write our set D_I defined above as

$$D_I = D_{\bigcup [a_n, b_n]} = \bigcup D_{[a_n, b_n]},$$

which is again a union of finite/countable sets, and the proof is complete. ■

Remark 3.16. *To be more explicit about the statement ⊖:*

- $\mathbb{R} = \bigcup_{k=1}^{\infty} [-k, k]$
- $(a, b) = \bigcup_{k=1}^{\infty} [a + \frac{b-a}{3k}, b - \frac{b-a}{3k}]$
- $(-\infty, b] = \bigcup_{k=1}^{\infty} [-k - |b|, b]$
- $(-\infty, b) = \bigcup_{k=1}^{\infty} [-k - |b|, b - \frac{1}{2k}]$
- ...

3.10 Continuous Inverse Theorem

↪ **Theorem 3.21**

Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a continuous function. Let $S := f(I)$. Suppose f strictly increasing. Then, for any $y \in S$, there is precisely one $x \in I$ s.t. $f(x) = y$.

Proof. Suppose x_1, x_2 s.t. $f(x_1) = f(x_2) = y$. f strictly increasing, hence both $x_1 > x_2$ and $x_1 < x_2$ are impossible, hence $x_1 = x_2$. ∎

↪ **Definition 3.16: Inverse**

et $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$ be a continuous function. Let $S := f(I)$. $\forall y \in S$, we set $g(y) = x \in I$ s.t. $f(x) = y$. This defines a function $g : S \to I$ s.t. $g(S) = I$. This gives

$$(f \circ g)(y) = y \,\forall y \in S; \quad (g \circ f)(x) = x \,\forall x \in I.$$

g is call the *inverse* of f; we often denote $g = f^{-1}$.

↪ **Proposition 3.7**

If f strictly increasing, so is f^{-1}.

↪ **Theorem 3.22: Continuous Inverse Theorem**

Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be strictly increasing and continuous. Then, $g = f^{-1}$ is also strictly increasing and continuous, on $S = f(I)$.

Proof. We show only continuous. Suppose g not continuous at some point $c \in S$; assume c not an endpoint, for now. Since g not continuous at c, we have that

$$j_g(c) = \lim_{y \to c^+} g(y) - \lim_{y \to c^-} g(y) > 0.$$

Let $x \in I$ s.t. $x \neq g(c)$ and s.t.

$$\lim_{y \to c^-} g(y) < x < \lim_{y \to c^+} g(y).$$

Then, there is no $y \in S$ s.t. $g(y) = x$, by our construction. But this contradicts the fact that $g(S) = I$, and hence g must be continuous on S, ■

4 Differentiation

4.1 Introduction

↪ **Definition 4.1: Differentiability**

Let $I \subseteq \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ and $c \in I$. We say that f is *differentiable* at c if the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists. If this limit exists, we denote it $f'(c)$ and call it the *derivative* of f at c.

↪ **Theorem 4.1**

If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.

Proof. We have for $x \in I \setminus \{c\}$,

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c}(x - c).$$

f being differentiable at c gives that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c),$$

so be algebraic properties of limits,

$$\lim_{x \to c}(f(x) - f(c)) = (\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \lim_{x \to c}(x - c)) = f'(c) \cdot 0 = 0,$$

hence, $\lim_{x\to c} f(x) = f(c)$, and thus f continuous at c. ■

Remark 4.1. *The converse of this theorem does* not *hold.*

⊛ **Example 4.1: Continuous $\not\Rightarrow$ differentiable**

Consider $f(x) = |x|$. This function is continuous on $\mathbb{R}$ but not differentiable at $c = 0$;

$$\frac{f(x)-f(c)}{x-c} = \frac{|x|}{x} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$
$$\implies \lim_{x\to c^+} \frac{f(x)-f(c)}{x-c} = 1, \lim_{x\to c^-} \frac{f(x)-f(c)}{x-c} = -1$$
$$\implies \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \text{ DNE } \implies f \text{ not differentiable at} c = 0.$$

↪ **Theorem 4.2: Algebraic Properties of the Derivative**

Let $I \subseteq \mathbb{R}$ be an interval and $c \in I$. Let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be differentiable at c. Then

1. For any $k \in \mathbb{R}$, kf differentiable at c, and moreover,
$$(kf)'(c) = k \cdot f'(c).$$
2. $f + g$ is differentiable at c;
$$(f+g)'(c) = f'(c) + g'(c).$$
3. (Product Rule) $f \cdot g$ is differentiable at c and
$$(fg)'(c) = f'(c) \cdot g(c) + f(c) \cdot g'(c)$$
4. (Quotient Rule) If $g(x) \neq 0 \, \forall x \in I$, then the quotient function $\frac{f}{g}$ is differentiable at c;
$$\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}$$

Proof. 1.

2.

3.

4. Let $h(x) = \frac{f(x)}{g(x)}$. Then,

$$\begin{aligned}
\frac{h(x)-h(c)}{x-c} &= \frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x-c} \\
&= \frac{f(x)g(c) - f(c)g(x)}{(x-c)(g(x)g(c))} \\
&= \frac{\overbrace{f(x)g(c) - f(c)g(c)} + \overbrace{f(c)g(c) - f(c)g(x)}}{(x-c)g(x)g(c)} \\
&= \frac{(f(x)-f(c))g(c)}{(x-c)g(x)g(c)} - \frac{(g(x)-g(c))f(c)}{(x-c)g(x)g(c)} \qquad \circledast \\
\lim_{x\to c} \circledast &= \lim_{x\to c} \frac{f'(c)g(c)}{g(x)g(c)} - \frac{g'(c)f(c)}{g(x)g(c)} \\
&= \frac{f'(c)g(c) - g'(c)f(c)}{[g(c)]^2}
\end{aligned}$$

■

↪ **Definition 4.2**

If f' exists on every point $c \in I$, then we say that f is differentiable on I. This gives a function

$$f' : I \to \mathbb{R}.$$

↪ **Proposition 4.1: Power Rule**

Let $f : I \to \mathbb{R}$, $f(x) = x^n, n \in \mathbb{N}$. We have that $f'(x) = nx^{n-1}$.

Proof. If $n = 1$, then $f = x$, and so $\frac{f(x)-f(c)}{x-c} = \frac{x-c}{x-c} = 1$. Suppose the rule holds up to some $n \in \mathbb{N}$. Consider $f = x^{n+1}$. Then,

$$\begin{aligned}
f(x) &= x^{n+1} = x^n x \\
\overset{\text{power rule}}{\Longrightarrow} f'(x) &= \underbrace{nx^{n-1}}_{\text{assumption}} \cdot x + x^n \\
&= (n+1)x^n
\end{aligned}$$

■

⊛ **Example 4.2**

Prove that $\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$ and $\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x$.

4.2 The Chain Rule

↪ **Theorem 4.3: Caratheodory Theorem**

Let I be an interval, $f : I \to \mathbb{R}$, and $c \in I$. TFAE:

1. f is differentiable at c;
2. $\exists$ a function $\varphi : I \to \mathbb{R}$, continuous at c, such that

$$f(x) = f(c) + \varphi(x)(x - c), \forall x \in I.$$

Remark 4.2. *From 2. $\implies$ 1., we have, moreover, that $f'(c) = \varphi(c)$.*

Proof. (1. $\implies$ 2.) Let

$$\varphi : I \to \mathbb{R}, x \mapsto \begin{cases} \frac{f(x)-f(c)}{x-c} & x \neq c \\ f'(c) & x = c \end{cases}.$$

We have, then,

$$\lim_{x\to c} \varphi(x) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c} = f'(c) = \varphi(c),$$

hence, φ is continuous at c. For $x \neq c$, the desired relation $f(x) = f(c) + \varphi(x)(x - c)$ holds by definition.

(1. $\impliedby$ 2.) If $x \neq c$, we have that $\frac{f(x)-f(c)}{x-c} = \varphi(x)$. Moreover, φ continuous at c, hence $\lim_{x\to c} \varphi(x) = \varphi(c)$, and thus $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists, and moreover, is equal to $\varphi(c)$. Thus, f differentiable at c and $f'(c) = \varphi(c)$. ■

↪ **Theorem 4.4: Chain Rule**

Let I, J be intervals in $\mathbb{R}$, and let $g : I \to \mathbb{R}$, $f : J \to \mathbb{R}$ be s.t. $f(J) \subseteq I$. Let $c \in J$; then, if f differentiable at c and g differentiable at $f(c)$, then the composite function

$$h = g \circ f, \quad h : J \to \mathbb{R},$$

is differentiable at c, and moreover,

$$h'(c) = (g \circ f)'(c) = g'(f(c)) \cdot f'(c)$$

Proof. Given $f'(c)$ exists, the Caratheodory theorem gives that there exists a function φ :

$J \to \mathbb{R}$ which is continuous at c such that

$$f(x) - f(c) = \varphi(x)(x - c) \ \forall\, x \in J.$$

Similarly, since g is differentiable at $f(c) \in I$, there exists a function $\Psi : I \to \mathbb{R}$ continuous at $f(c)$, such that

$$g(y) - g(f(c)) = \Psi(y)(y - c).$$

Letting $y = f(x)$, this yields

$$\begin{aligned} g(f(x)) - g(f(c)) &= \Psi(f(x))(f(x) - c) \\ &= \Psi(f(x))\varphi(x)(x - c). \end{aligned}$$

Letting $h = g \circ f$ and $r(x) = \Psi(f(x))\varphi(x)$ gives us

$$h(x) - h(c) = r(x)(x - c)\, \forall\, x \in J.$$

By compositions, r is continuous at c, and moreover, $r(c) = \Psi(f(c))\varphi(c) = g'(f(c))f'(c)$, and hence,

$$h'(c) = g'(f(c)) \cdot f'(c).$$

■

4.3 Derivative of the Inverse Function

↪ **Theorem 4.5**

Let I be an interval and $f : I \to \mathbb{R}$ be a strictly increasing continuous function. Let $J = f(I)$ and $g : J \to \mathbb{R}$ be the inverse of f. Suppose f differentiable at c, $f'(c) \neq 0$. Then, g is differentiable at $f(c)$, and $g'(f(c)) = \frac{1}{f'(c)}$.

Proof. By the Caratheodory theorem, we have some $\varphi : I \to \mathbb{R}$ continuous at c s.t. $f(x) - f(c) = \varphi(x)(x - c)$, where $\varphi(c) = f'(c)$. Since $f'(c) \neq 0$ and φ continuous at c, we have that there exists $\delta > 0$ s.t. $\varphi(x) \neq 0\, \forall\, x \in (c - \delta, c + \delta) \cap I$. ■

5 Appendix

5.1 Interesting Results

A summary of theorems or results that stemmed from assignments, tutorials, etc..

↪ **Theorem 5.1: Cesàro Summation**

Consider a convergent sequence (x_n). Then, the sequence defined

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n} \sum_{k=1}^{n} x_k$$

is also convergent, and we have that

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n.$$

↪ **Theorem 5.2: Stolz-Cesàro**

Let (y_n) be a strictly monotone sequence of positive numbers. Consider some other sequence (x_n). We have, then, if

$$\lim_{n \to \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L$$

exists, then the limit

$$\lim_{n \to \infty} \frac{x_n}{y_n} = L$$

as well.

↪ **Lemma 5.1: Fekete's Subadditive Lemma**

A sequence (x_n) is called *subadditive* if $\forall\, n, m \in \mathbb{N}$,

$$x_{n+m} \leq x_n + x_m$$

holds. For any subadditive sequence (x_n), its limit exists, and moreover,

$$\lim_{n \to \infty} x_n = \inf\{\frac{x_n}{n} : n \in \mathbb{N}\}.$$

↪ Definition 5.1: Lacunary Sequence

A sequence x_n is called *lacunary* if there exists some real number q such that $\forall n \in \mathbb{N}$,

$$\frac{x_{n+1}}{x_n} \geq q > 1.$$