MATH454 - Analysis 3

Summary

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1 Sigma-Algebras and Measures

Definition 1 (σ-algebra): A σ-algebra of subsets of a space X is a collection $\mathcal{F}$ of subsets of X satisfying

- $X \in \mathcal{F}$;
- $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$;
- $\{A_n\}_{n=1}^{\infty} \subset \mathcal{F} \Rightarrow \bigcup_{n\geq 1} A_n \in \mathcal{F}$.

Some σ-algebras can be "generated" by a collection $\mathcal{C}$, in which case we denote $\mathcal{F} = \sigma(\mathcal{C})$, being the smallest σ-algebra containing $\mathcal{C}$. In general generators are not unique

A canonical example is the Borel σ-algebra,

$$\mathfrak{B}_{\mathbb{R}} = \sigma(\{A \subset \mathbb{R} : A \text{ open}\}).$$

Definition 2 (Measure): A measure $\mu : \mathcal{F} \to [0, \infty]$ is a set function defined on a σ-algebra satisfying

- $\mu(\varnothing) = 0$;
- for $\{A_n\} \subseteq \mathcal{F}$ disjoint, $\mu\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} \mu(A_n)$.

Definition 3 (Lebesgue Outer Measure): For all $A \subseteq \mathbb{R}$,

$$m^*(A) \coloneqq \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : I_n \text{ open intervals s.t.} \bigcup_{n\geq 1} I_n \supseteq A\right\}.$$

A set is then called *Lebesgue measurable* if for every $B \subseteq \mathbb{R}$,

$$m^*(B) = m^*(A \cap B) + m^*(A^c \cap B).$$

Theorem 1: Let $\mathcal{M} = \{A \subseteq \mathbb{R} : A \text{ Lebesgue measurable}\}$. Then, $\mathcal{M}$ a σ-algebra, and $m \coloneqq m^*|_{\mathcal{M}}$ is a measure on $\mathcal{M}$.

Theorem 2: $m, \mathcal{M}$ is translation invariant, $m((a, b)) = b - a$, $\mathfrak{B}_{\mathbb{R}} \subsetneq \mathcal{M}$, outer regular ($m(A) = \inf\{m(G) : G \text{ open}, G \supseteq A\}$), and inner regular ($m(A) = \sup\{m(K) : K \text{ compact}, K \subseteq A\}$).

Theorem 3: $\mathcal{M}$ is complete, and $\mathcal{M} = \overline{\mathfrak{B}_{\mathbb{R}}}$.

Theorem 4: m is the unique measure on $\mathfrak{B}_{\mathbb{R}}$ that is finite on compact sets and translation invariant, up to rescaling.

Theorem 5: A collection of subsets of X, $\mathcal{I}$, is called a *π-system* if $A, B \in \mathcal{I} \Rightarrow A \cap B \in \mathcal{I}$. A collection of subsets of X, $\mathcal{D}$, is called a *d-system* if $X \in \mathcal{D}$, $A \subseteq B \in \mathcal{D} \Rightarrow B \setminus A \in \mathcal{D}$, and $A_n \uparrow\in \mathcal{D} \Rightarrow \bigcup_n (A_n) \in \mathcal{D}$.

Let $\mathcal{I}$ be a collection of subsets of X and let $d(\mathcal{I})$ be the smallest d-system containing $\mathcal{I}$. Then, $d(\mathcal{I}) = \sigma(\mathcal{I})$.

Theorem 6: There exists

- an uncountable set of measure 0 (the Cantor set);
- a non-measurable set (the Vitali set);
- a set that is Lebesgue but not Borel measurable.

2 Integration Theory

Definition 4: A function $f : \mathbb{R} \to \overline{\mathbb{R}}$ is called (Lebesgue) measurable if for every $a \in \mathbb{R}$, $\{f < a\} := f^{-1}([-\infty, a)) \in \mathcal{M}$.

If f, g measurable, so are $f \pm g, f \cdot g, c \cdot f, \min\{f, g\}, \max\{f, g\}, f^+, f^-$. If $\{f_n\}$ a sequence of measurable functions, $\limsup_n f_n$, $\liminf_n f_n$, etc are all measurable.

Definition 5 (Integral): A simple function is of the form $\varphi = \sum_{k=1}^{L} a_k \mathbb{1}_{\{A_k\}}$ for measurable sets A_k, and $L < \infty$. We define

$$\int_{\mathbb{R}} \varphi := \sum_{k=1}^{L} a_k m(A_k).$$

For any $f \geq 0$, we can find a sequence of simple functions that increase to f. Let f be a nonnegative measurable function. We define

$$\int_{\mathbb{R}} f := \sup\left\{ \int_{\mathbb{R}} \varphi : \varphi \leq f \right\}.$$

Finally, for general f measurable, we define

$$\int_{\mathbb{R}} f := \int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-.$$

We say a function f *integrable* and write $f \in L^1(\mathbb{R})$ if $\int_{\mathbb{R}} |f| < \infty$.

Definition 6 (Convergence a.e., in measure): Let $\{f_n\}$ be a sequence of measurable functions. We say $f_n \to f$ *almost everywhere* on $\mathbb{R}$ if $f_n(x) \to f(x)$ for almost every $x \in \mathbb{R}$. We say $f_n \to f$ *in measure* if for every $\delta > 0$, $m\{|f_n - f| > \delta\} \to 0$.

Theorem 7: $f_n \to f$ a.e. $\Rightarrow f_n \to f$ in measure.

$f_n \to f$ in measure $\Rightarrow f_{n_k} \to f$ a.e. along some subsequence $\{n_k\}$.

Theorem 8 (Egorov's): Let $A \in \mathcal{M}$ be a finite measure set such that $f_n \to f$ a.e. on A. Then, for every $\varepsilon > 0$, there is a closed set $A_\varepsilon \subseteq A$ such that $m(A \setminus A_\varepsilon) \leq \varepsilon$ and $f_n \to f$ uniformly on A_ε.

Theorem 9 (Lusin's): Let $A \in \mathcal{M}$ be a finite measure set and f measurable. For every $\varepsilon > 0$, there exists a closed set $A_\varepsilon \subseteq A$ such that $m(A \setminus A_\varepsilon) \leq \varepsilon$ and $f|_{A_\varepsilon}$ continuous on A_ε.

Theorem 10 (Monotone Convergence): $f_n \uparrow f$, nonnegative, $\Rightarrow \int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$.

Theorem 11 (Fatou): $\int_{\mathbb{R}} \liminf_n f_n \leq \liminf_n \int_{\mathbb{R}} f_n$.

Theorem 12 (Dominated Convergence): $f_n \to f$ a.e. and exists $g \in L^1(\mathbb{R})$ such that $|f_n| \leq |g|$, then $\int_{\mathbb{R}} |f_n - f| \to 0$.

Definition 7 (L^p): Put $\|f\|_p := \left(\int_{\mathbb{R}} |f|^p\right)^{\frac{1}{p}}$ and $L^p(\mathbb{R}) = \left\{f \text{ measurable} : \|f\|_p < \infty\right\}$.

Put also $\|f\|_\infty = \inf\left\{a \in \overline{\mathbb{R}} : |f| \leq a \text{ a.e.}\right\}$, and $L^\infty = \{f : \|f\|_\infty < \infty\}$.

Theorem 13 (Holder, Minkowski): $\|fg\|_1 \leq \|f\|_p \|g\|_q$ where $\frac{1}{p} + \frac{1}{q} = 1$ and $f, g \in L^p, L^q$ resp.

$\|f + g\|_p \leq \|f\|_p + \|g\|_p$.

Theorem 14: L^p a complete space with respect to the L^p norm, $\|\cdot\|_p$

Theorem 15: $C_c(\mathbb{R})$ dense in L^p for $p < \infty$

Theorem 16: A sequence of functions $\{f_n\}$ is said to be *uniformly integrable* on a set A if

$$\lim_{M \to \infty} \sup_n \left(\int_{A \cap \{|f_n| > M\}} |f_n|\right) = 0.$$

Suppose $f_n, f \in L^1(A)$ for $m(A) < \infty$. Then, $f_n \to f$ in L^1 if and only if $\{f_n\}$ uniformly integrable and $f_n \to f$ in measure on A.

3 Product Space

Definition 8: Define $\mathcal{M}^2 = \sigma(\{A \times B : A, B \in \mathcal{M}\})$. For $E \in \mathcal{M}^2$, define $E_x = \{y \in (x, y) \in E\}$, with a symmetric definition for E^y.

Theorem 17: $\int_{\mathbb{R}} m(E_x) \, \mathrm{d}x = \int_{\mathbb{R}} m(E^y) \, \mathrm{d}y$. As such, define the measure of a set $E \in \mathcal{M}^2$ by

$$m(E) := \int_{\mathbb{R}} m(E_x) \, \mathrm{d}x.$$

Theorem 18 (Tonelli's): Let $f \geq 0 : \mathbb{R}^2 \to \overline{\mathbb{R}}$ be $\mathcal{M}^2$-measurable. Then,

$$\int_{\mathbb{R}^2} f = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y) \, \mathrm{d}x \right) \mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y) \, \mathrm{d}y \right) \mathrm{d}x.$$

Theorem 19 (Fubini's): Let $f \in L^1(\mathbb{R}^2)$. Then, the above statement also holds.

4 Differentiation

Theorem 20 (Lebesgue Differentiation Theorem): Let $f \in L^1(\mathbb{R})$. For $x \in \mathbb{R}$, let $\{I_n\}$ be a sequence of open intervals such that $x \in I_n$ for every $n \geq 1$, and $m(I_n) \to 0$. Then, for almost every $x \in \mathbb{R}$,

$$\lim_{n \to \infty} \frac{1}{m(I_n)} \int_{I_n} |f(t) - f(x)| \, \mathrm{d}x = 0.$$

Theorem 21: Suppose F nondecreasing on $[a, b]$. Then, F' exists a.e., $F' \in L^1([a, b])$, and $\int_a^b F' \leq F(b) - F(a)$.

Definition 9 (Bounded Variation): A function $f : [a, b] \to \mathbb{R}$ is of *bounded variation* if

$$T_F(a, b) := \sup\left\{ \sum_{k=1}^{N} |f(x_k) - f(x_{k-1})| : a = x_0 < \cdots < x_N = b \right\} < \infty.$$

We write $F \in \mathrm{BV}([a, b])$.

Theorem 22: $F \in \mathrm{BV}([a, b]) \Leftrightarrow F = H - G$ where H, G increasing.

Definition 10 (Absolutely Continuous): A function F is *absolutely continuous* on $[a, b]$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $\{(a_k, b_k)\}_{k=1}^N$ a disjoint sequence of open intervals with $\sum_{k=1}^N (b_k - a_k) \leq \delta$, then $\sum_{k=1}^N |F(b_k) - F(a_k)| \leq \varepsilon$. We write $F \in \mathrm{AC}([a, b])$.

Theorem 23 (FTC): $F \in \mathrm{AC}([a, b])$, then F' exists almost everywhere, $F' \in L^1([a, b])$, and

$$F(x) - F(a) = \int_a^x F'(t) \, \mathrm{d}t \; \forall \, x \in [a, b].$$