

MATH454 - Analysis 3

Measure spaces; Integration.

Based on lectures from Fall 2024 by Prof. Linan Chen.

Notes by Louis Meunier

Contents

1 Sigma Algebras and Measures	3
1.1 A Review of Riemann Integration	3
1.2 Sigma Algebras	3
1.3 Measures	5
1.4 Constructing the Lebesgue Measure on \mathbb{R}	7
1.5 Lebesgue-Measurable Sets	10
1.6 Properties of the Lebesgue Measure	12
1.7 Relationship between $\mathfrak{B}_{\mathbb{R}}$ and \mathcal{M}	15
1.8 Some Special Sets	16
1.8.1 Uncountable Null Set?	16
1.8.2 Non-Measurable Sets?	18
1.8.3 Non-Borel Measurable Set?	21
2 Integration Theory	21
2.1 Measurable Functions	21
2.2 Approximation by Simple Functions	27
2.3 Convergence Almost Everywhere vs Convergence in Measure	31
2.4 Egorov's Theorem and Lusin's Theorem	33
2.5 Construction of Integrals	35
2.5.1 Integral of Simple Functions	35
2.5.2 Integral of Non-Negative Functions	37
2.5.3 Integral of General Measurable, Integrable Functions	39
2.6 Convergence Theorems of Integral	41
2.7 Riemann Integral vs Lebesgue Integral	46
2.8 L^p -space	48
2.8.1 Dense Subspaces of $L^p(\mathbb{R})$	51
2.9 Convergence Modes and Uniform Integrability	53
3 Product space	56
3.1 Preparations	56
3.2 Product Lebesgue σ -Algebra	57
3.3 Product Measure	59
3.4 Fubini's Theorem	63
4 Differentiation	66

4.1 Hardy-Littlewood Maximal Function	67
4.2 Lebesgue Differentiation Theorem	69

§1 SIGMA ALGEBRAS AND MEASURES

§1.1 A Review of Riemann Integration

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $[a, b] \subset \mathbb{R}$. Define a **partition** of $[a, b]$ as the set

$$\text{part}([a, b]) := \{a =: x_0 < x_1 < \dots < x_N =: b\}.$$

We can then define the upper and lower Riemann integrals of f over the region $[a, b]$ as

$$\begin{aligned} \text{upper: } \int_a^{\overline{b}} f(x) \, dx &:= \inf_{\text{part}([a, b])} \left\{ \sum_{i=1}^N \sup_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \right\} \\ \text{lower: } \int_a^{\underline{b}} f(x) \, dx &:= \sup_{\text{part}([a, b])} \left\{ \sum_{i=1}^N \inf_{x \in [x_{i-1}, x_i]} f(x) \cdot (x_i - x_{i-1}) \right\}. \end{aligned}$$

We then say f **Riemann integrable** if these two quantities are equal, and denote this value by $\int_a^b f(x) \, dx$.

Many “nice-enough” (continuous, monotonic, etc.) functions are Riemann integrable, but many that we would like to be able to “integrate” are simply not, for instance Dirichlet’s function $x \mapsto \begin{cases} 1 & x \in \mathbb{Q} \cap [a, b] \\ 0 & x \in \mathbb{Q}^c \cap [a, b] \end{cases}$. Hence, we need a more general notion of integration.

§1.2 Sigma Algebras

↪ **Definition 1.1** (Sigma algebra): Let X be a *space* (a nonempty set) and \mathcal{F} a collection of subsets of X . \mathcal{F} a *sigma algebra* or simply σ -algebra of X if the following hold:

1. $X \in \mathcal{F}$
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ (closed under complement)
3. $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ (closed under countable unions)

↪ **Proposition 1.1:**

4. $\emptyset \in \mathcal{F}$
5. $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{F} \Rightarrow \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
6. $A_1, \dots, A_n \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n, \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$
7. $A, B \in \mathcal{F} \Rightarrow A \setminus B, B \setminus A \in \mathcal{F}$

⊗ **Example 1.1:** The “largest” sigma algebra of a set X is the power set 2^X , the smallest the trivial $\{\emptyset, X\}$.

Given a set $A \subset X$, the set $\mathcal{F}_A := \{\emptyset, X, A, A^c\}$ is a sigma algebra; given two disjoint sets $A, B \subset X$, then $\mathcal{F}_{A, B} := \{\emptyset, X, A, A^c, B, B^c, A \cup B, A^c \cap B^c\}$ a sigma algebra.

↪ **Definition 1.2** (Generating a sigma algebra): Let X be a nonempty set, and \mathcal{C} a collection of subsets of X . Then, the σ -algebra *generated* by \mathcal{C} , denoted $\sigma(\mathcal{C})$, is such that

1. $\sigma(\mathcal{C})$ a sigma algebra with $\mathcal{C} \subseteq \sigma(\mathcal{C})$
2. if \mathcal{F}' a sigma algebra with $\mathcal{C} \subseteq \mathcal{F}'$, then $\mathcal{F}' \supseteq \sigma(\mathcal{C})$

Namely, $\sigma(\mathcal{C})$ is the smallest sigma algebra “containing” (as a subset) \mathcal{C} .

↪ **Proposition 1.2:**

1. $\sigma(\mathcal{C}) = \bigcap \{ \mathcal{F} : \mathcal{F} \text{ a sigma algebra containing } \mathcal{C} \}$
2. if \mathcal{C} itself a sigma algebra, then $\sigma(\mathcal{C}) = \mathcal{C}$
3. if $\mathcal{C}_1, \mathcal{C}_2$ are two collections of subsets of X such that $\mathcal{C}_1 \subseteq \mathcal{C}_2$, then $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}_2)$

↪ **Definition 1.3** (The Borel sigma-algebra): The *Borel σ -algebra*, denoted $\mathfrak{B}_{\mathbb{R}}$, on the real line is given by

$$\mathfrak{B}_{\mathbb{R}} := \sigma(\{\text{open subsets of } \mathbb{R}\}).$$

We call sets in $\mathfrak{B}_{\mathbb{R}}$ *Borel sets*.

↪ **Proposition 1.3:** $\mathfrak{B}_{\mathbb{R}}$ is also generated by the sets

- $\{(a, b) : a < b \in \mathbb{R}\}$
- $\{(a, b] : a < b \in \mathbb{R}\}$
- $\{[a, b] : a < b \in \mathbb{R}\}$
- $\{[a, b) : a < b \in \mathbb{R}\} \circledast$
- $\{(-\infty, c) : c \in \mathbb{R}\}$
- $\{(-\infty, c] : c \in \mathbb{R}\}$
- etc.

PROOF. We prove just \circledast . It suffices to show that the generating sets of each σ -algebra is contained in the other σ -algebra. Let $a < b \in \mathbb{R}$. Then,

$$(a, b) = \bigcup_{n=1}^{\infty} \underbrace{\left[a + \frac{1}{n}, b \right)}_{\in \circledast} \in \sigma(\{[a, b)\}) \Rightarrow \mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[a, b)\}).$$

Conversely,

$$[a, b) = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b \right) \in \mathfrak{B}_{\mathbb{R}}.$$

■

↪ **Proposition 1.4:** All intervals (open, closed, half open, half closed, finite, etc) are Borel sets; any set obtained from countable set operations of intervals are Borel; all singletons are Borel; any finite and countable sets are Borel.

§1.3 Measures

↪ **Definition 1.4** (Measurable Space): Let X be a space and \mathcal{F} a σ -algebra. We call the tuple (X, \mathcal{F}) a *measurable space*.

↪ **Definition 1.5** (Measure): Let (X, \mathcal{F}) be a measurable space. A *measure* is a function $\mu : \mathcal{F} \rightarrow [0, \infty]$ satisfying

- (i) $\mu(\emptyset) = 0$;
- (ii) if $\{A_n\} \subseteq \mathcal{F}$ a sequence of (pairwise) disjoint sets, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n),$$

i.e. μ is *countably additive*. We further call μ

- *finite* if $\mu(X) < \infty$,
- a *probability measure* if $\mu(X) = 1$,
- σ -*finite* if $\exists \{A_n\} \subseteq \mathcal{F}$ such that $X = \bigcup_{n=1}^{\infty} A_n$ with $\mu(A_n) < \infty \forall n \geq 1$,

and call the triple (X, \mathcal{F}, μ) a *measure space*.

⊕ **Example 1.2:** The measure on $\mathfrak{B}_{\mathbb{R}}$ given by

$$A \mapsto \begin{cases} |A| & \text{if } A \text{ finite} \\ \infty & \text{else} \end{cases}$$

is called the *counting measure*.

Fix $x_0 \in \mathbb{R}$, then the measure on $\mathfrak{B}_{\mathbb{R}}$ given by

$$A \mapsto \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{else} \end{cases}$$

is called the *point mass at x_0* .

↪ **Theorem 1.1** (Properties of Measures): Fix a measure space (X, \mathcal{F}, μ) . The following properties hold:

1. (finite additivity) For any sequence $\{A_n\}_{n=1}^N \subseteq \mathcal{F}$ of disjoint sets,

$$\mu\left(\bigcup_{n=1}^N A_n\right) = \sum_{n=1}^N \mu(A_n).$$

2. (monotonicity) For any $A \subseteq B \in \mathcal{F}$, then $\mu(A) \leq \mu(B)$.
3. (countable/finite subadditivity) For any sequence $\{A_n\} \subseteq \mathcal{F}$ (**not necessarily disjoint**),

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n),$$

an analogous statement holding for a finite collection of sets A_1, \dots, A_N .

4. (continuity from below) For $\{A_n\} \subseteq \mathcal{F}$ such that $A_n \subseteq A_{n+1} \forall n \geq 1$ (in which case we say $\{A_n\}$ “increasing” and write $A_n \uparrow$) we have

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

5. (continuity from above) For $\{A_n\} \subseteq \mathcal{F}, A_n \supseteq A_{n+1} \forall n \geq 1$ (we write $A_n \downarrow$) we have that if $\mu(A_1) < \infty$,

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

Remark 1.1: In 4., note that since A_n increasing, that the union $\bigcup_{n=1}^{\infty} A_n \supseteq A_m$ for any arbitrarily large m ; indeed, one could logically right $\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} A_n$. In this notation, then, 4. simply states that we may interchange limit and measure. A similar argument can be viewed for 5. (how?).

Remark 1.2: The finiteness condition in 5. may be slightly modified such as to state that $\mu(A_n) < \infty$ for some n ; remark why this would suffice to ensure the entire rest of the sequence has finite measure.

PROOF.

1. Extend A_1, \dots, A_N to an infinite sequence by $A_n := \emptyset$ for $n > N$. Then this simply follows from countable additivity and $\mu(\emptyset) = 0$.
2. We may write $B = A \cup (B \setminus A)$; this is a disjoint union of sets. By finite additivity, then,

$$\mu(B) = \mu(A) + \mu(B \setminus A) \geq \mu(A),$$

since the measure is positive.

3. We prove only for a countable union; use the technique from 1. to extend to finite. We first “disjointify” the sequence such that we can use the countable additivity

axiom. Let $B_1 = A_1, B_n = A_n \setminus \left(\bigcup_{i=1}^{n-1} A_i\right)$ for $n \geq 2$. Remark then that $\{B_n\} \subseteq \mathcal{F}$ is a disjoint sequence of sets, and that $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$. By countable additivity and subadditivity,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \leq \sum_{n=1}^{\infty} \mu(A_n).$$

4. We again “disjointify” the sequence $\{A_n\}$. Put $B_1 = A_1, B_n = A_n \setminus A_{n-1}$ for all $n \geq 2$ (remark that this is equivalent to the construction from the previous proof because the sets are increasing). Then, again, $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$, and in particular, for all $N \geq 1, \bigcup_{n=1}^N B_n = A_N$. Then

$$\begin{aligned} \mu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \\ &= \lim_{N \rightarrow \infty} \sum_{n=1}^N \mu(B_n) \\ &= \lim_{N \rightarrow \infty} \mu\left(\bigcup_{n=1}^N B_n\right) = \lim_{N \rightarrow \infty} \mu(A_N). \end{aligned}$$

5. We yet again disjointify, backwards (in a way) from the previous case. Put $B_n = A_1 \setminus A_n$ for all $n \geq 1$. Then, $\{B_n\} \subseteq \mathcal{F}, B_n$ increasing, and $\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \bigcap_{n=1}^{\infty} A_n$. Then, by continuity from below,

$$\mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n \rightarrow \infty} \mu(B_n) = \lim_{n \rightarrow \infty} \mu(A_1 \setminus A_n)$$

and also

$$\begin{aligned} \mu(A_1) &= \mu\left(A_1 \setminus \bigcap_{n=1}^{\infty} A_n\right) + \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \\ &= \mu(A_1 \setminus A_n) + \mu(A_n), \end{aligned}$$

and combining these two equalities yields the desired result. ■

§1.4 Constructing the Lebesgue Measure on \mathbb{R}

↪ **Definition 1.6** (Lebesgue outer measure): For all $A \subseteq \mathbb{R}$, define

$$m^*(A) := \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n, I_n \text{ open intervals} \right\},$$

called the *Lebesgue outer measure* of A (where $\ell(I)$ is the length of interval I , i.e. the absolute value of the difference of its endpoints, if finite, or ∞ if not).

↪ **Proposition 1.5:** The following properties of m^* hold:

1. $m^*(A) \geq 0$ for all $A \subseteq \mathbb{R}$, and $m^*(\emptyset) = 0$.
2. (monotonicity) For $A \subseteq B$, $m^*(A) \leq m^*(B)$.
3. (countable subadditivity) For $\{A_n\}, A_n \subseteq \mathbb{R}$, $m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n)$.¹
4. If $I \subseteq \mathbb{R}$ an interval, then $m^*(I) = \ell(I)$.
5. m^* is translation invariant; for any $A \subseteq \mathbb{R}, x \in \mathbb{R}$, $m^*(A) = m^*(A + x)$ where $A + x := \{a + x : a \in A\}$.
6. For all $A \subseteq \mathbb{R}$, $m^*(A) = \inf\{m^*(B) : A \subseteq B \subseteq \mathbb{R}, B \text{ open}\}$.
7. If $A = A_1 \cup A_2 \subseteq \mathbb{R}$ with $d(A_1, A_2) > 0$,² then $m^*(A_1) + m^*(A_2) = m^*(A)$.
8. If $A = \bigcup_{k=1}^{\infty} J_k$ where J_k 's are "almost disjoint intervals" (i.e. share at most endpoints), then $m^*(A) = \sum_{k=1}^{\infty} m^*(J_k) = \sum_{k=1}^{\infty} \ell(J_k)$.

PROOF.

3. If $m^*(A_n) = \infty$, for any n , we are done, so assume wlog $m^*(A_n) < \infty$ for all n . Then, for each n and $\varepsilon > 0$, one can choose open intervals $\{I_{n,i}\}_{i \geq 1}$ such that $A_n \subseteq \bigcup_{i=1}^{\infty} I_{n,i}$ and $\sum_{i=1}^{\infty} \ell(I_{n,i}) \leq m^*(A_n) + \frac{\varepsilon}{2^n}$. Hence

$$\begin{aligned} \bigcup_{n=1}^{\infty} A_n &\subseteq \bigcup_{n=1, i=1}^{\infty} I_{n,i} \\ \Rightarrow m^*\left(\bigcup_{n=1}^{\infty} A_n\right) &\leq \sum_{n,i=1}^{\infty} \ell(I_{n,i}) = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \ell(I_{n,i}) \leq \sum_{n=1}^{\infty} \left(m^*(A_n) + \frac{\varepsilon}{2^n}\right) = \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon, \end{aligned}$$

and as ε arbitrary, the statement follows.

4. We prove first for $I = [a, b]$. For any $\varepsilon > 0$, set $I_1 = (a - \varepsilon, b + \varepsilon)$; then $I \subseteq I_1$ so $m^*(I) \leq \ell(I_1) = (b - 1) + 2\varepsilon$ hence $m^*(I) \leq b - a = \ell(I)$. Conversely, let $\{I_n\}$ be any open-interval covering of I (wlog, each of finite length; else the statement holds trivially). Since I compact, it can be covered by finitely many of the I_n 's, say $\{I_n\}_{n=1}^N$, denoting $I_n = (a_n, b_n)$ (with relabelling, etc). Moreover, we can pick the a_n, b_n 's such that $a_1 < a, b_N > b$, and generally $a_n < b_{n-1} \forall 2 \leq n \leq N$. Then,

$$\begin{aligned} \sum_{n=1}^{\infty} \ell(I_n) &\geq \sum_{n=1}^N \ell(I_n) = b_1 - a_1 + \sum_{n=2}^N (b_n - a_n) \\ &\geq b_1 - a_1 + \sum_{n=2}^N (b_n - b_{n-1}) \\ &= b_N - a_1 \geq b - 1 = \ell(I), \end{aligned}$$

hence since the cover was arbitrary, $m^*(A) \geq \ell(I)$, and equality holds.

Now, suppose I finite, with endpoints $a < b$. Then for any $\frac{b-a}{2} > \varepsilon > 0$, then

$$[a + \varepsilon, b - \varepsilon] \subseteq I \subseteq [a - \varepsilon, b + \varepsilon],$$

¹More generally, any set function on $2^{\mathbb{R}}$ that satisfies 1., 2., and 3. is called an *outer measure*.

²Remark: this is a stronger requirement than disjointness!

hence by monotonicity and the previous part of this proof

$$m^*([a + \varepsilon, b - \varepsilon]) = b - a - 2\varepsilon \leq m^*(I) \leq b - a + 2\varepsilon = m^*([a - \varepsilon, b + \varepsilon]),$$

from which it follows that $m^*(I) = b - a = \ell(I)$.

Finally, suppose I infinite. Then, $\forall M \geq 0, \exists$ closed, finite interval I_M with $I_M \subseteq I$ and $\ell(I_M) \geq M$. Hence, $m^*(I) \geq m^*(I_M) \geq M$ and thus as M arbitrary it must be that $m^*(I) = \infty = \ell(I)$.

6. Denote $\tilde{m}(A) := \inf\{m^*(B) : A \subseteq B \subseteq \mathbb{R}, B \text{ open}\}$. For any $A \subseteq B \subseteq \mathbb{R}$ with B open, monotonicity gives that $m^*(A) \leq m^*(B)$, hence $m^*(A) \leq \tilde{m}(A)$. Conversely, assuming wlog $m^*(A) < \infty$ (else holds trivially), then for all $\varepsilon > 0$, there exists $\{I_n\}$ such that $A \subseteq \bigcup_{n=1}^{\infty} I_n$ with $\sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon$. Setting $B := \bigcup_{n=1}^{\infty} I_n$, we have that $A \subseteq B$ and $m^*(B) = m^*(\bigcup I_n) \leq$ (by finite subadditivity) $\sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} \ell(I_n) \leq m^*(A) + \varepsilon$ hence $m^*(B) \leq m^*(A)$ for all B . Thus $m^*(A) \geq \tilde{m}(A)$ and equality holds.

7. Put $\delta := d(A_1, A_2) > 0$. Clearly $m^*(A) \leq m^*(A_1) + m^*(A_2)$ by finite subadditivity. wlog, $m^*(A) < \infty$ (and hence $m^*(A_i) < \infty, i = 1, 2$) (else holds trivially). Then $\forall \varepsilon > 0, \exists \{I_n\} : A \subseteq \bigcup I_n$ and $\sum \ell(I_n) \leq m^*(A) + \varepsilon$. Then, for all n , we consider a "refinement" of I_n ; namely, let $\{I_{n,i}\}_{i \geq 1}$ such that $I_n \subseteq \bigcup_i I_{n,i}$ and $\ell(I_{n,i}) < \delta$ and $\sum_i \ell(I_{n,i}) \leq \ell(I_n) + \frac{\varepsilon}{2^n}$. Relabel $\{I_{n,i} : n, i \geq 1\} \rightsquigarrow \{J_m : m \geq 1\}$ (both are countable). Then, $\{J_m\}$ defines an open-interval cover of A , and since $\ell(J_m) < \delta$ for each m, J_m intersects at most one A_i . For each m and $p = 1, 2$, put

$$M_p := \{m : J_m \cap A_p \neq \emptyset\},$$

noting that $M_1 \cap M_2 = \emptyset$. Then $\{J_m : m \in M_p\}$ is an open covering of A_p , and so

$$\begin{aligned} m^*(A_1) + m^*(A_2) &\leq \sum_{m \in M_1} \ell(J_m) + \sum_{m \in M_2} \ell(J_m) \\ &\leq \sum_{m=1}^{\infty} \ell(J_m) = \sum_{n,i=1}^{\infty} \ell(I_{n,i}) \\ &\leq \sum_n \left(\ell(I_n) + \frac{\varepsilon}{2^n} \right) \\ &= \sum_n \ell(I_n) + \varepsilon \\ &\leq m^*(A) + 2\varepsilon, \end{aligned}$$

and hence equality follows.

8. If $\ell(J_k) = \infty$ for some k , then since $J_k \subseteq A$, subadditivity gives us that $m^*(J_k) \leq m^*(A)$ and so $m^*(A) = \infty = \sum_{k=1}^{\infty} \ell(J_k)$ (since if any J_k infinite, the sum of the lengths of all of them will also be infinite).

Suppose then $\ell(J_k) < \infty$ for all k . Fix $\varepsilon > 0$. Then for all $k \geq 1$, choose $I_k \subseteq J_k$ such that $\ell(J_k) \leq \ell(I_k) + \frac{\varepsilon}{2^k}$. For any $N \geq 1$, we can choose a subset $\{I_1, \dots, I_N\}$ of intervals such that all are disjoint, with strictly positive distance between them, and so

$$\begin{aligned}
\bigcup_{k=1}^N I_k &\subseteq \bigcup_{k=1}^N I_k \subseteq A \\
\Rightarrow m^*(A) &\geq m^*\left(\bigcup_{k=1}^N I_k\right) \geq \sum_{k=1}^N \ell(I_k) \\
&\geq \sum_{k=1}^N \left(\ell(J_k) - \frac{\varepsilon}{2^k}\right) \\
&\geq \sum_{k=1}^N \ell(J_k) - \varepsilon \\
\Rightarrow m^*(A) &\geq \sum_{k=1}^{\infty} \ell(J_k),
\end{aligned}$$

the second inequality following from finite subadditivity. The converse of the final inequality holds trivially. ■

§1.5 Lebesgue-Measurable Sets

↪ **Definition 1.7:** $A \subseteq \mathbb{R}$ is m^* -measurable if $\forall B \subseteq \mathbb{R}$,

$$m^*(B) = m^*(B \cap A) + m^*(B \cap A^c).$$

Remark 1.3: By subadditivity, \leq always holds in the definition above.

↪ **Theorem 1.2** (Carathéodary's Theorem): Let

$$\mathcal{M} := \{A \subseteq \mathbb{R} : A \text{ } m^* \text{-measurable}\}.$$

Then, \mathcal{M} is a σ -algebra of subsets of \mathbb{R} .

Define $m : \mathcal{M} \rightarrow [0, \infty]$, $m(A) = m^*(A)$. Then, m is a measure on \mathcal{M} , called the *Lebesgue measure* on \mathbb{R} . We call sets in \mathcal{M} *Lebesgue-measurable* or simply *measurable* (if clear from context) accordingly. We call $(\mathbb{R}, \mathcal{M}, m)$ the *Lebesgue measure space*.

PROOF. The first two σ -algebra axioms are easy. We have for any $B \subseteq \mathbb{R}$ that

$$m^*(B \cap \mathbb{R}) + m^*(B \cap \mathbb{R}^c) = m^*(B) + m^*(B \cap \emptyset) = m^*(B)$$

so $\mathbb{R} \in \mathcal{M}$. Further, $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$ by the symmetry of the requirement for sets to be in \mathcal{M} .

The final axiom takes more work. We show first \mathcal{M} closed under finite unions; by induction it suffices to show for 2 sets. Let $A_1, A_2 \in \mathcal{M}$. Then, for all $B \subseteq \mathbb{R}$,

$$\begin{aligned}
m^*(B) &= m^*(B \cap A_1) + m^*(B \cap A_1^c) \\
&= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*(B \cap A_1^c \cap A_2^c) \\
&= m^*(B \cap A_1) + m^*(B \cap A_1^c \cap A_2) + m^*(B \cap (A_1 \cup A_2)^c)
\end{aligned}$$

Note that $(B \cap A_1) \cup (B \cap A_1^c \cap A_2) = B \cap (A_1 \cup A_2)$, hence by subadditivity,

$$m^*(B) \geq m^*(B \cap (A_1 \cup A_2)) + m^*(B \cap (A_1 \cup A_2)^c),$$

and since the other direction of the inequality comes for free, we conclude $A_1 \cup A_2 \in \mathcal{M}$.

Let now $\{A_n\} \subseteq \mathcal{M}$. We “disjointify” $\{A_n\}$; put $B_1 := A_1$, $B_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i$, $n \geq 2$, noting $\bigcup_n A_n = \bigcup_n B_n$, and each $B_n \in \mathcal{M}$, as each is but a finite number of set operations applied to the A_n 's, and thus in \mathcal{M} as demonstrated above. Put $E_n := \bigcup_{i=1}^n B_i$, noting again $E_n \in \mathcal{M}$. Then, for all $B \subseteq \mathbb{R}$,

$$\begin{aligned} m^*(B) &= m^*\left(\underbrace{B \cap E_n}_{\text{chop up } B_n}\right) + m^*\left(\underbrace{B \cap E_n^c}_{E_n \subseteq \cup B_n \Rightarrow E_n^c \supseteq (\cup B_n)^c}\right) \\ &\geq m^*\left(B \cap \underbrace{E_n \cap B_n}_{=B_n}\right) + m^*\left(B \cap \underbrace{E_n \cap B_n^c}_{=E_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\ &\geq m^*(B \cap B_n) + m^*\left(\underbrace{B \cap E_{n-1}}_{\text{chop up } B_{n-1}}\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\ &\geq m^*(B \cap B_n) + m^*(B \cap E_{n-1} \cap B_{n-1}) \\ &\quad + m^*(B \cap E_{n-1} \cap B_{n-1}^c) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right). \end{aligned}$$

Notice that the last line is essentially the second applied to B_{n-1} ; hence, we have a repeating (essentially, “descending”) pattern in this manner, which we repeat until $n \rightarrow 1$. We have, thus, that

$$m^*(B) \geq \sum_{i=1}^n [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right),$$

so taking $n \rightarrow \infty$,

$$\begin{aligned} m^*(B) &\geq \sum_{i=1}^{\infty} [m^*(B \cap B_i)] + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right) \\ &\geq m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)\right) + m^*\left(B \cap \left(\bigcup_{n=1}^{\infty} B_n\right)^c\right). \end{aligned}$$

As usual, the inverse inequality comes for free, and thus we can conclude $\bigcup_{n=1}^{\infty} B_n$ also m^* -measurable, and thus so is $\bigcup_{n=1}^{\infty} A_n$. This proves \mathcal{M} a σ -algebra.

We show now m a measure. By previous propositions, we have that $m \geq 0$ and $m(\emptyset) = 0$ (since $m = m^*|_{\mathcal{M}}$), so it remains to prove countable subadditivity.

Let $\{A_n\} \subseteq \mathcal{M}$ -disjoint. Following precisely the same argument as above, used to prove that \mathcal{M} closed under countable unions, shows that for any $n \geq 1$

$$m\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n m(A_i),$$

that is, finite additivity holds, and thus by subadditivity

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq m\left(\bigcup_{i=1}^n A_i\right) = \sum_{i=1}^n m(A_i),$$

and so taking the limit of $n \rightarrow \infty$, we have

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} m(A_i),$$

with the converse inequality coming for free. Thus, m indeed a measure on \mathcal{M} . ■

↪ **Proposition 1.6:** \mathcal{M}, m translation invariant; for all $A \in \mathcal{M}, x \in \mathbb{R}, x + A = \{x + a : a \in A\} \in \mathcal{M}$ and $m(A) = m(A + x)$.

Remark 1.4: We would like this to hold, heuristically, since if we shift sets on the real line, we should expect their length to remain constant.

PROOF. For all $B \subseteq \mathbb{R}$, we have (since m^* translation invariant)

$$\begin{aligned} m^*(B) &= m^*(B - x) = m^*\left(\underbrace{(B - x) \cap A}_{=B \cap (A+x)}\right) + m^*\left(\underbrace{(B - x) \cap A^c}_{=B \cap (A^c+x)=B \cap (A+x)^c}\right) \\ &= m^*(B \cap (A + x)) + m^*(B \cap (A + x)^c), \end{aligned}$$

thus $A + x \in \mathcal{M}$, and since m^* translation invariant, it follows that m is. ■

↪ **Theorem 1.3:** $\forall a, b \in \mathbb{R}$ with $a < b$, $(a, b) \in \mathcal{M}$, and $m((a, b)) = b - a$.

Remark 1.5: Again, we'd like this to hold, heuristically, since we would like the measure of an interval to simply be its length; we'd moreover like to be able to measure intervals, i.e. have intervals be contained in \mathcal{M} .

↪ **Corollary 1.1:** $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M}$

PROOF. $\mathfrak{B}_{\mathbb{R}}$ is generated by open intervals of the form (a, b) . All such intervals are in \mathcal{M} by the previous theorem, and hence the proof. ■

§1.6 Properties of the Lebesgue Measure

↪ **Proposition 1.7** (Regularity Properties of m): For all $A \in \mathcal{M}$, the following hold.

- For all $\varepsilon > 0$, $\exists G$ open such that $A \subseteq G$ and $m(G \setminus A) < \varepsilon$.
- For all $\varepsilon > 0$, $\exists F$ -closed such that $F \subseteq A$ and $m(A \setminus F) \leq \varepsilon$.
- $m(A) = \inf\{m(G) : G \text{ open}, G \supseteq A\}$.
- $m(A) = \sup\{m(K) : K \text{ compact}, K \subseteq A\}$.
- If $m(A) < \infty$, then for all $\varepsilon > 0$, $\exists K \subseteq A$ compact, such that $m(A \setminus K) < \varepsilon$.
- If $m(A) < \infty$, then for all $\varepsilon \geq 0$, \exists finite collection of open intervals I_1, \dots, I_N such that $m\left(A \Delta \left(\bigcup_{n=1}^N I_n\right)\right) \leq \varepsilon$.

↪ **Proposition 1.8** (Completeness of m): $(\mathbb{R}, \mathcal{M}, m)$ is complete, in the sense that for all $A \subseteq \mathbb{R}$, if $\exists B \in \mathcal{M}$ such that $A \subseteq B$ and $m(B) = 0$, then $A \in \mathcal{M}$ and $m(A) = 0$.

Equivalently, any subset of a null set is again a null set.

Remark 1.6: In general, $A \in \mathcal{F}, B \subseteq A \not\Rightarrow B \in \mathcal{F}$.

↪ **Proposition 1.9:** Up to rescaling, m is the unique, nontrivial measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ that is finite on compact sets and is translation invariant, i.e. if μ another such measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ with $\mu = c \cdot m$ for $c > 0$, then $\mu = m$.

Remark 1.7: Such a c is simply $c = \mu((0, 1))$.

To prove this proposition, we first introduce some helpful tooling:

↪ **Theorem 1.4** (Dynkin's π -d): Given a space X , let \mathcal{C} be a collection of subsets of X . \mathcal{C} is called a π -system if $A, B \in \mathcal{C} \Rightarrow A \cap B \in \mathcal{C}$ (that is, it is closed under finite intersections).

Let $\mathcal{F} = \sigma(\mathcal{C})$, and suppose μ_1, μ_2 are two finite measures on (X, \mathcal{F}) such that $\mu_1(X) = \mu_2(X)$ and $\mu_1 = \mu_2$ when restricted to \mathcal{C} . Then, $\mu_1 = \mu_2$ on all of \mathcal{F} .

↪ **Proposition 1.10:** $\{\emptyset\} \cup \{(a, b) : a < b \in \mathbb{R}\}$ a π -system.

↪ **Proposition 1.11:** If μ a measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ such that for all intervals I , $\mu(I) = \ell(I)$, then $\mu = m$.

PROOF. Consider for all $n \geq 1$ $\mu|_{\mathfrak{B}_{[-n, n]}}$. Clearly, $\mu([-n, n]) = m([-n, n]) = 2n$, and for all $a, b \in \mathbb{R}$, $\mu((a, b) \cap [-n, n]) = \ell((a, b) \cap [-n, n]) = m((a, b) \cap [-n, n])$. Thus, by the previous theorem, μ must match m on all of $\mathfrak{B}_{[-n, n]}$.

Let now $A \in \mathfrak{B}_{\mathbb{R}}$. Let $A_n := A \cap [-n, n] \in \mathfrak{B}_{[-n, n]}$. By continuity of m from below,

$$\begin{aligned}\mu(A) &= \lim_{n \rightarrow \infty} \mu(A_n) \\ &= \lim_{n \rightarrow \infty} m(A_n) \\ &= m(A),\end{aligned}$$

hence $\mu = m$. ■

↪ **Proposition 1.12:** If μ a measure on $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}})$ assigning finite values to compact sets and is translation invariant, then $\mu = cm$ for some $c > 0$.

Remark 1.8: This proposition is also tacitly stating that $\mathfrak{B}_{\mathbb{R}}$ translation invariant; this needs to be shown.

↪ **Lemma 1.1:** $\mathfrak{B}_{\mathbb{R}}$ translation invariant; for any $A \in \mathfrak{B}_{\mathbb{R}}, x \in \mathbb{R}, A + x \in \mathfrak{B}_{\mathbb{R}}$.

PROOF. We employ the “good set strategy”; fix some $x \in \mathbb{R}$ and let

$$\Sigma := \{B \in \mathfrak{B}_{\mathbb{R}} : B + x \in \mathfrak{B}_{\mathbb{R}}\}.$$

We have by construction $\Sigma \subseteq \mathfrak{B}_{\mathbb{R}}$. One can check too that Σ a σ -algebra. But in addition, its easy to see that $\{(a, b) : a < b \in \mathbb{R}\} \subseteq \Sigma$, since a translated interval is just another interval, and since these sets generate $\mathfrak{B}_{\mathbb{R}}$, it must be further that $\mathfrak{B}_{\mathbb{R}} \subseteq \Sigma$, completing the proof. ■

PROOF. (of the proposition) Let $c = \mu((0, 1])$, noting that $c > 0$ (why? Consider what would happen if $c = 0$).

This implies that $\forall n \geq 1, \mu\left(\left(0, \frac{1}{n}\right]\right) = \frac{c}{n}$ (obtained by “chopping up” $(0, 1]$ into n disjoint intervals); from here we can draw many further conclusions:

$$\forall m = 1, \dots, n - 1, \mu\left(\left(0, \frac{m}{n}\right]\right) = \frac{m}{n}c$$

$$\Rightarrow \forall q \in \mathbb{Q} \cap (0, 1], \mu((0, q]) = qc$$

$$\Rightarrow \forall q \in \mathbb{Q}^+, \mu((0, q]) = q \cdot c \text{ (translate)}$$

$$\Rightarrow \forall a \in \mathbb{R}, \mu((a, a + q]) = q \cdot c$$

$$\Rightarrow \forall \text{intervals } I, \mu(I) = c \cdot \ell(I) \text{ (continuity)}$$

$$\Rightarrow \forall n \geq 1, a, b \in \mathbb{R}, \mu((a, b) \cap [-n, n]) = c \cdot \ell((a, b) \cap [-n, n]) = c \cdot m((a, b) \cap [-n, n]),$$

but then, $\mu = c \cdot m$ on $\mathfrak{B}_{\mathbb{R}[-n, n]}$, and by appealing again the Dynkin’s, $\mu = c \cdot m$ on all of $\mathfrak{B}_{\mathbb{R}}$. ■

↪ **Proposition 1.13** (Scaling): m has the *scaling property* that $\forall A \in \mathcal{M}, c \in \mathbb{R}, c \cdot A = \{cx : x \in A\} \in \mathcal{M}$, and $m(c \cdot A) = |c| m(A)$.

PROOF. Assume $c \neq 0$. Given $A \subseteq \mathbb{R}$, remark that $\{I_n\}$ an open interval cover of A iff $\{cI_n\}$ and open interval cover of cA , and $\ell(cI_n) = |c| \ell(I_n)$, and thus $m^*(cA) = |c| m^*(A)$.

Now, suppose $A \in \mathcal{M}$. Then, we have for any $B \subseteq \mathbb{R}$,

$$\begin{aligned} m^*(B) &= |c| m^*\left(\frac{1}{c}B\right) = |c| m^*\left(\frac{1}{c}B \cap A\right) + |c| m^*\left(\frac{1}{c}B \cap A^c\right) \\ &= m^*(B \cap cA) + m^*(B \cap (cA)^c), \end{aligned}$$

so $cA \in \mathcal{M}$. ■

§1.7 Relationship between $\mathfrak{B}_{\mathbb{R}}$ and \mathcal{M}

↪ **Definition 1.8**: Given (X, \mathcal{F}, μ) , consider the following collection of subsets of X ,

$$\mathcal{N} := \{B \subseteq X : \exists A \in \mathcal{F} \text{ s.t. } \mu(A) = 0, B \subseteq A\}.$$

Put $\overline{\mathcal{F}} := \sigma(\mathcal{F} \cup \mathcal{N})$; this is called the *completion* of \mathcal{F} with respect to μ .

↪ **Proposition 1.14**: $\overline{\mathcal{F}} = \{F \subseteq X : \exists E, G \in \mathcal{F} \text{ s.t. } \exists E \subseteq F \subseteq G \text{ and } m(G \setminus E) = 0\}$.

PROOF. Put \mathcal{G} the set on the right; one can check \mathcal{G} a σ -algebra. Since $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{N} \subseteq \mathcal{G}$, we have $\overline{\mathcal{F}} \subseteq \mathcal{G}$.

Conversely, for any $F \in \mathcal{G}$, we have $E, G \in \mathcal{F}$ such that $E \subseteq F \subseteq G$ with $m(G \setminus E) = 0$. We can rewrite

$$F = \underbrace{E}_{\in \mathcal{F}} \cup \underbrace{(F \setminus E)}_{\substack{\subseteq G \setminus E \\ \Rightarrow \mu(F \setminus E) = 0 \\ \Rightarrow G \setminus E \in \mathcal{N}}},$$

hence $F \in \mathcal{F} \cup \mathcal{N}$ and thus in $\overline{\mathcal{F}}$, and equality holds. ■

↪ **Definition 1.9**: Given (X, \mathcal{F}, μ) , μ can be *extended* to $\overline{\mathcal{F}}$ by, for each $F \in \overline{\mathcal{F}}$ with $E \subseteq F \subseteq G$ s.t. $\mu(G \setminus E) = 0$, put

$$\mu(F) = \mu(E) = \mu(G).$$

We call then $(X, \overline{\mathcal{F}}, \mu)$ a *complete measure space*.

Remark 1.9: It isn't obvious that this is well defined a priori; in particular, the E, G sets are certainly not guaranteed to be unique in general, so one must check that this definition is valid regardless of choice of "sandwich sets".

↪ **Theorem 1.5:** $(\mathbb{R}, \mathcal{M}, m)$ is the completion of $(\mathbb{R}, \mathfrak{B}_{\mathbb{R}}, m)$.

PROOF. Given $A \in \mathcal{M}$, then $\forall n \geq 1, \exists G_n$ -open with $A \subseteq G_n$ s.t. $m^*(G_n \setminus A) \leq \frac{1}{n}$ and $\exists F_n$ -closed with $F_n \subseteq A$ s.t. $m^*(A \setminus F_n) \leq \frac{1}{n}$.

Put $C := \bigcap_{n=1}^{\infty} G_n, B := \bigcap_{n=1}^{\infty} F_n$, remarking that $C, B \in \mathfrak{B}_{\mathbb{R}}, B \subseteq A \subseteq C$, and moreover

$$\begin{aligned} m(C \setminus A) &\leq \frac{1}{n}, m(A \setminus B) \leq \frac{1}{n} \\ \Rightarrow m(C \setminus B) &= m(C \setminus A) + m(A \setminus B) \leq \frac{2}{n}, \end{aligned}$$

but n can be arbitrarily large, hence $m(C \setminus B) = 0$; in short, given a measurable set, we can “sandwich it” arbitrarily closely with Borel sets. Thus, $A \in \overline{\mathfrak{B}_{\mathbb{R}}} \Rightarrow \mathcal{M} \subseteq \overline{\mathfrak{B}_{\mathbb{R}}}$. But recall that \mathcal{M} complete, so $\mathfrak{B}_{\mathbb{R}} \subseteq \mathcal{M} \Rightarrow \overline{\mathfrak{B}_{\mathbb{R}}} \subseteq \overline{\mathcal{M}} = \mathcal{M}$, and thus $\overline{\mathfrak{B}_{\mathbb{R}}} = \mathcal{M}$ indeed.

Heuristically, this means that any measurable set is “different” from a Borel set by at most a null set. ■

§1.8 Some Special Sets

1.8.1 Uncountable Null Set?

Remark that for any countable set $A \in \mathcal{M}, m(A) = 0$; indeed, one may write $A = \bigcup_{n=1}^{\infty} \{a_n\}$ for singleton sets $\{a_n\}$, and so

$$m(A) = \sum_{n=1}^{\infty} m(a_n) = 0.$$

One naturally asks the opposite question, does there exist a measurable, *uncountable* set with measure 0? We construct a particular one here, the Cantor set, C .

This requires an “inductive” construction. Define $C_0 = [0, 1]$, and define C_k to be C_{k-1} after removing the middle third from each of its disjoint components. For instance $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$, then $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$, and so on. This may be clearest graphically:



Remark that the $C_n \downarrow$. Put finally

$$C := \bigcap_{n=1}^{\infty} C_n.$$

↪ **Proposition 1.15:** The following hold for the Cantor set C :

1. C is closed (and thus $C \in \mathfrak{B}_{\mathbb{R}}$);
2. $m(C) = 0$;
3. C is uncountable.

PROOF.

1. For each n , C_n is the countable (indeed, finite) union of 2^n -many disjoint, closed intervals, hence each C_n closed. C is thus a countable intersection of closed sets, and is thus itself closed.
2. For each n , each of the 2^n disjoint closed intervals in C_n has length $\frac{1}{3^n}$, hence

$$m(C_n) = \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n.$$

Since $\{C_n\} \downarrow$, by continuity of m we have

$$m(C) = \lim_{n \rightarrow \infty} m(C_n) = \lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = 0.$$

3. This part is a little trickier. Notice that for any $x \in [0, 1]$, we can define a sequence (a_n) where each $a_n \in \{0, 1, 2\}$, and such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n};$$

in particular, this is just the base-3 representation of x , which we denote $(x)_3 = (a_1 a_2 \dots)$.

I claim now that

$$C = \{x \in [0, 1] : (x)_3 \text{ has no 1's}\}.$$

Indeed, at each stage n of the construction of the Cantor set, we get rid of the segment of the real line that would correspond to the $a_n = 1$. One should note that $(x)_3$ not necessarily unique; for instance $\left(\frac{1}{3}\right)_3 = (1, 0, 0, \dots) = (0, 2, 2, \dots)$, but if we specifically consider all x such that there *exists* a base three representation with no 1's, i.e. like $\frac{1}{3}$, then C indeed captures all the desired numbers.

Thus, we have that

$$\text{card}(C) = \text{card}(\{(a_n) : a_n = 0, 2\}).$$

Define now the function

$$f : C \rightarrow [0, 1], \quad x \mapsto \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot \frac{1}{2^n}, \text{ where } (x)_3 = (a_n)$$

i.e., we “squish” the base-3 representation into a base-2 representation of a number.

This is surjective; for any $y \in [0, 1]$, $(b_n) := (y)_2$ contains only 0's and 1's, hence $(2b_n)$

contains only 0's and 1's, so let x be the number such that $(x)_3 = (2b_n)$. This necessarily exists, indeed, we simply take our definitions backwards:

$$x := \sum_{n=1}^{\infty} \frac{2b_n}{3^n},$$

which maps to y under f and is contained in C . Hence, $\text{card}(C) \geq \text{card}([0, 1])$; but $[0, 1]$ uncountable, and thus so is C . ■

We can naturally extend the function f used here to map the entire interval $[0, 1] \rightarrow [0, 1]$ as follows

$$f(x) := \begin{cases} \sum_{n=1}^{\infty} \frac{a_n}{2} \cdot \frac{1}{2^n} & \text{if } x \in C, (x)_3 = (a_n) \\ f(a) & \text{if } x \notin C \text{ then } x \in (a, b) \text{ s.t. } (a, b) \text{ removed from } [0, 1] \end{cases}$$

This function is often called the *Devil's Staircase* or *Cantor-Lebesgue function*.

↔ **Proposition 1.16:**

1. $f(0) = 0, f(1) = 1, f \equiv \frac{1}{2}$ on $(\frac{1}{3}, \frac{2}{3}), f \equiv \frac{1}{4}$ on $(\frac{1}{9}, \frac{2}{9})$
2. $f : [0, 1] \rightarrow [0, 1]$ a surjection
3. f is nondecreasing
4. f is continuous

PROOF. 1., 2., clear from construction.

For 3., let $x_1 < x_2 \in C$, and suppose $(x_1)_3 = (a_n), (x_2)_3 = (b_n)$. Then, since $x_1 < x_2$, it must be that a_n, b_n can only be equal up to some finite N ; then the next $0 = a_{N+1} < b_{N+1} = 2$. Hence, it follows that the "modified binary expansion" that arises from f gives directly that $f(x_1) \leq f(x_2)$.

For 4., f is clearly continuous on $[0, 1] - C$, since it is piecewise-constant here. Also, f is "one-sided continuous" at each of the "boundary points" $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \dots$. If $x \in C$, for any $n \geq 1$, there must be x_n, x_n' such that $x_n < x < x_n'$ (if $x = 0$, only need x_n' , if $x = 1$, only need x_n) and $f(x_n') - f(x_n) \leq \frac{1}{2^n}$. Then, f is continuous at x by monotonicity of f . ■

1.8.2 Non-Measurable Sets?

We've shown then that there is indeed an uncountable set of measure 0. Another question we may ask ourselves is, is there a $A \subseteq \mathbb{R}$ that is non-measurable? The answer to this turns out to be yes, but the construction requires invoking the axiom of choice:

Axiom 1 (Of Choice): If Σ a collection of nonempty sets, then \exists a function

$$S : \Sigma \rightarrow \bigcup_{A \in \Sigma} A,$$

such that $A \in \sigma, S(A) \in A$. Such a function is called a *selection function*, and $S(A)$ a *representative* of A .

We construct now a non-measurable set, assuming the above. Consider $[0, 1]$, and define an equivalence relation \sim on $[0, 1]$ by

$$a \sim b \Leftrightarrow a - b \in \mathbb{Q}.$$

Its easy to check that this is indeed an equivalence relation. Denote by E_a the equivalence class containing a , and set $\Sigma = \{E_a : a \in [0, 1]\}$. Note that for any $E_a \in \Sigma, E_a \neq \emptyset$.

Invoking the axiom of choice, we can select exactly one element S_a from E_a for each $E_a \in \Sigma$. Set

$$N := \{S_a : S_a \text{ is a representative of } E_a, E_a \in \Sigma\}.$$

Proposition 1.17: N , called a *Vitali set*, is non-measurable.

PROOF. Assume towards a contradiction that N indeed measurable, $N \in \mathcal{M}$. Consider $[-1, 1] \cap \mathbb{Q}$; this is countable, so we can enumerate it $\{q_k\}, k \geq 1$. For each k , put

$$N_k := N + q_k.$$

By the assumption of measurability and translation invariance of m , it must be that each N_k measurable and has the same measure as N .

We claim each N_k disjoint. Assume not, then $\exists k \neq \ell$ (i.e. $q_k \neq q_\ell$) and $S_a, S_b \in N$ such that $S_a + q_k = S_b + q_\ell$. But then $S_a - S_b = q_\ell - q_k \in \mathbb{Q}$, hence $S_a \sim S_b$. But we constructed N to have only one representative from each equivalence class, hence it must be that $S_a = S_b$, and so $S_a + q_k = S_a + q_\ell \Rightarrow q_k = q_\ell$, contradicting the assumed distinctness of the q 's; hence, the N_k 's indeed disjoint.

We claim next that $[0, 1] \subseteq \bigcup_{k=1}^{\infty} N_k$. Let $x \in [0, 1]$. Then, $x \sim S_a$ for some unique $S_a \in N$ and so $x - S_a \in \mathbb{Q}$. But also, $x, S_a \in [0, 1]$, hence $x - S_a \in [-1, 1]$ (moreover, $x - S_a \in [-1, 1] \cap \mathbb{Q}$) and there must exist a k such that $x - S_a = q_k$, since the q_k 's enumerate the entire $[-1, 1] \cap \mathbb{Q}$. Thus, $x \in N_k$ by the construction of the N_k 's. Thus, $[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k$ indeed.

On the other hand, $\bigcup_{k=1}^{\infty} N_k \subseteq [-1, 2]$ and so we have the "bound"

$$[0, 1] \subseteq \bigcup_{n=1}^{\infty} N_k \subseteq [-1, 2].$$

Taking the measure of all sides then, we have the bound

$$1 \leq \mu\left(\bigcup_{n=1}^{\infty} N_k\right) \leq 3.$$

Invoking the disjointness of the N_k 's, we can also use countable additivity to write

$$\mu\left(\bigcup_{n=1}^{\infty} N_k\right) = \sum_{k=1}^{\infty} m(N_k) = \sum_{k=1}^{\infty} m(N),$$

but this final line is a sequence of positive, constant real numbers; hence, it is impossible for it to be within 1 and 3, and we have a contradiction. Hence, N indeed not measurable.

Remark that this proof also shows that $m^*(N_k) > 0$ so $m^*(N) > 0$ (given the interval bound on N we've found). ■

↪ **Proposition 1.18:** For every $A \in \mathcal{M}$ such that $m(A) > 0$, there exists $B \subseteq A$ such that B is non-measurable.

PROOF. Assume otherwise, that there is a $A \in \mathcal{M}$ with $m(A) > 0$ such that any subset B of A is also measurable.

Remark that $A \subseteq \bigcup_{n \in \mathbb{Z}} A \cap [n, n+1]$. Then, there exists an n such that $m(A \cap [n, n+1]) > 0$ and thus, translating $A' := A \cap [n, n+1] - n$, $m(A') > 0$, noting that $A' \subseteq [0, 1]$. Now, for any $B' \subseteq A'$, $B' + n \subseteq A$. By assumption, then $B' + n$ must be measurable so B' measurable.

In summary, then, we have $A' \subseteq [0, 1]$ with $m(A') > 0$ such that (by assumption) B' measurable for all $B' \subseteq A'$.

Let $N, \{q_k\}, N_k$ be as in the previous proof. Set

$$A_k' := A' \cap N_k, k \geq 1.$$

Then, A_k' disjoint, and

$$A' = [0, 1] \cap A' \subseteq \bigcup_{k=1}^{\infty} (N_k \cap A') = \bigcup_{k=1}^{\infty} A_k'.$$

Since $m(A') > 0$, there exists a k such that $m(A_k') > 0$. Set, for this k ,

$$L := \{\ell \geq 1 : q_\ell + q_k \in [-1, 1]\}.$$

This set is again countably infinite. We translate, obtaining a disjoint sequence of sets $\{q_\ell + A_k' : \ell \in L\}$; since $q_\ell + q_k \in [-1, 1] \cap \mathbb{Q}$, then $q_\ell + q_k = q_m$ for some unique m , and so $q_\ell + A_k' = q_\ell + A' \cap (N + q_k) \subseteq N_m$. Hence, we have on the one hand that by countable additivity

$$\bigcup_{\ell \in L} (q_\ell + A_k') \subseteq [-1, 2] \Rightarrow \sum_{\ell \in L} m(q_\ell + A_k') \leq 3,$$

and so it must be that $m(q_\ell + A_k') = m(A_k') = 0$ (else the series couldn't be finite), contradicting the finiteness assumption on $m(A_k')$. ■

1.8.3 Non-Borel Measurable Set?

We may ask, is there $A \in \mathcal{M}$ such that $A \notin \mathfrak{B}_{\mathbb{R}}$?

Let $f : [0, 1] \rightarrow [0, 1]$ be the Cantor-Lebesgue function, and put $g(x) = f(x) + x$; note that g is continuous and strictly increasing, and is defined $g : [0, 1] \rightarrow [0, 2]$. Remark that g bijective; the strictly increasing gives injective, and moreover $g(0) = 0, g(1) = 2$ hence by intermediate value theorem it is surjective. Hence, $g^{-1} : [0, 2] \rightarrow [0, 1]$ exists, and is also continuous, so in short g is a homeomorphism; it maps open to open, closed to closed. In particular, if $A \in \mathfrak{B}_{\mathbb{R}}$, then $g(A) \in \mathfrak{B}_{\mathbb{R}}$.

Recall that if (a, b) an open interval that gets removed from the construction of C , then f is constant and so g will map (a, b) to another open interval of the same length $b - a$. Thus,

$$m(g([0, 1] \setminus C)) = m([0, 1] \setminus C) = 1.$$

Hence, $m(g(C)) = 2 - 1 = 1 > 0$, since $g(C \cup [0, 1] \setminus C) = [0, 2]$. Hence, there exists a $B \subseteq g(C)$ such that $B \notin \mathcal{M}$, as per the previous proposition.

Let $A := g^{-1}(B)$; then $A \subseteq g^{-1}(g(C)) = C$. Since $m(C) = 0, A \in \mathcal{M}$ and $m(A) = 0$. But, $A \notin \mathfrak{B}_{\mathbb{R}}$; if it were, then $g(A) = B \in \mathfrak{B}_{\mathbb{R}}$, since g "maintains" Borel sets, but B is not even Lebesgue measurable and so this is a contradiction).

§2 INTEGRATION THEORY

§2.1 Measurable Functions

We will be considering functions f defined on \mathbb{R} or some subset of \mathbb{R} that could take positive or negative infinity as its value i.e.

$$f : \mathbb{R} \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\},$$

where $\overline{\mathbb{R}}$ the *extended real line*; we say f is $\overline{\mathbb{R}}$ -valued. If f never takes $\infty, -\infty$ for any $x \in \mathbb{R}$, we say f finite-valued, or just \mathbb{R} -valued.

For all $a \in \mathbb{R}$, we consider inverse images

$$f^{-1}([-\infty, a]) := \{x \in \mathbb{R} : f(x) \in [-\infty, a]\} = \{f < a\},$$

remarking the inclusion of $-\infty$; similarly

$$f^{-1}((a, \infty]) := \{x \in \mathbb{R} : f(x) \in (a, \infty]\} = \{f > a\},$$

and so on, for any $B \subseteq \mathbb{R}$,

$$f^{-1}(B) := \{x \in \mathbb{R} : f(x) \in B\} = \{f \in B\}.$$

Remark that

$$\begin{aligned} f^{-1}(B^c) &= (f^{-1}(B))^c \\ f^{-1}(A \cap B) &= f^{-1}(A) \cap f^{-1}(B) \\ f^{-1}(A \cup B) &= f^{-1}(A) \cup f^{-1}(B), \end{aligned}$$

which extend naturally for countable unions/intersections.

↪ **Definition 2.1** (Measurable Function): $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is measurable if $\forall a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) \in \mathcal{M}.$$

↪ **Proposition 2.1** (Equivalent Definitions of Measurability):

$$\begin{aligned} f \text{ is measurable} &\Leftrightarrow \forall a \in \mathbb{R}, f^{-1}([a, \infty]) \in \mathcal{M} \\ &\Leftrightarrow \forall a \in \mathbb{R}, f^{-1}((a, \infty]) \in \mathcal{M} \\ &\Leftrightarrow \forall a \in \mathbb{R}, f^{-1}([-\infty, a]) \in \mathcal{M} \end{aligned}$$

PROOF. We prove just the last equivalence. Notice that $\forall a \in \mathbb{R}$, we can use the commuting of inverse images with countable unions, intersections, complement to write

$$f^{-1}([-\infty, a)) = \bigcup_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a - \frac{1}{n}\right)\right)$$

and

$$f^{-1}([-\infty, a]) = \bigcap_{n=1}^{\infty} f^{-1}\left(\left[-\infty, a + \frac{1}{n}\right)\right).$$

■

↪ **Proposition 2.2**: If f finite-valued, Then

$$\begin{aligned} f \text{ is measurable} &\Leftrightarrow \forall a < b \in \mathbb{R}, f^{-1}((a, b)) \in \mathcal{M} \\ &\Leftrightarrow \dots f^{-1}((a, b]) \in \mathcal{M} \\ &\Leftrightarrow \dots f^{-1}([a, b)) \in \mathcal{M} \\ &\Leftrightarrow \dots f^{-1}([a, b]) \in \mathcal{M}. \end{aligned}$$

↪ **Definition 2.2** (Extended Borel Sigma Algebra): Define the Borel “extended” algebra $\mathfrak{B}_{\overline{\mathbb{R}}}$ of subsets of $\overline{\mathbb{R}}$, defined by

$$\mathfrak{B}_{\overline{\mathbb{R}}} := \sigma(\mathfrak{B}_{\mathbb{R}} \cup \{-\infty\}, \{\infty\}).$$

↪ **Proposition 2.3**: $\mathfrak{B}_{\overline{\mathbb{R}}} = \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$.

PROOF. For every $a \in \mathbb{R}$, we may write

$$[-\infty, a) = \underbrace{(-\infty, a)}_{\in \mathfrak{B}_{\mathbb{R}}} \cup \{-\infty\} \in \mathfrak{B}_{\mathbb{R}},$$

so $\sigma(\{[-\infty, a) : a \in \mathbb{R}\}) \subseteq \mathfrak{B}_{\mathbb{R}}$.

Conversely, notice that

$$\{-\infty\} = \bigcap_{n=1}^{\infty} [-\infty, -n),$$

and

$$\{\infty\} = \overline{\mathbb{R}} - \left(\bigcup_{n=1}^{\infty} [-\infty, n) \right),$$

so $\{-\infty\}, \{\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$. Hence, for any $a \in \mathbb{R}$,

$$(-\infty, a) = [-\infty, a) - \{-\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\}),$$

and so $\mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$. $\{-\infty\}, \{\infty\} \in \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$ already, and thus $\mathfrak{B}_{\mathbb{R}} \subseteq \sigma(\{[-\infty, a) : a \in \mathbb{R}\})$. ■

↪ **Proposition 2.4:** $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ measurable \Leftrightarrow for all $B \in \mathfrak{B}_{\overline{\mathbb{R}}}, f^{-1}(B) \in \mathcal{M}$.

PROOF. \Leftarrow is immediate. For \Rightarrow , let C be a collection of subsets of $\overline{\mathbb{R}}$, then put

$$f^{-1}(C) := \{f^{-1}(B) : B \in C\}.$$

By an assignment question (2.6),

$$f^{-1}(\sigma(C)) = \sigma(f^{-1}(C)).$$

Take $C = \{[-\infty, a) : a \in \mathbb{R}\}$. Then,

$$f^{-1}(\sigma(C)) = f^{-1}(\mathfrak{B}_{\overline{\mathbb{R}}}) = \sigma(f^{-1}(\{[-\infty, a) : a \in \mathbb{R}\})).$$

But f measurable, so $f^{-1}([- \infty, a)) \in \mathcal{M}$ for each $a \in \mathbb{R}$, hence $\sigma(f^{-1}(\{[-\infty, a) : a \in \mathbb{R}\})) \subseteq \mathcal{M}$ and so $f^{-1}(\sigma(C)) \subseteq \mathcal{M}$ completing the proof. ■

↪ **Corollary 2.1:** If f finite-valued, then f is measurable \Leftrightarrow for every $B \in \mathfrak{B}_{\mathbb{R}}, f^{-1}(B) \in \mathcal{M}$.

↪ **Proposition 2.5:** Given $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$, define the *finite valued component* of f given by

$$f_{\mathbb{R}}(x) := \begin{cases} f(x) & : -\infty < f(x) < \infty \\ 0 & \text{otherwise} \end{cases}.$$

Then, f measurable $\Leftrightarrow \forall B \in \mathfrak{B}_{\mathbb{R}}, f_{\mathbb{R}}^{-1}(B) \in \mathcal{M}$ AND $\{f = \infty\}, \{f = -\infty\}$ both in \mathcal{M} .

PROOF. (\Leftarrow) For any $a \in \mathbb{R}$,

$$f^{-1}([- \infty, a)) = \{f = -\infty\} \cup f^{-1}((-\infty, a)) = \{f = -\infty\} \cup f_{\mathbb{R}}^{-1}((-\infty, a)),$$

a union of measurable sets and hence is itself measurable.

(\Rightarrow) Remark that $\{f = \infty\}, \{f = -\infty\} \in \mathcal{M}$ automatically. For any $B \in \mathfrak{B}_{\mathbb{R}}$, we have $f_{\mathbb{R}}^{-1}(B) = \{x \in \mathbb{R} : f_{\mathbb{R}}(x) \in B\} = \{x \in \mathbb{R} : f(x) \in B, -\infty < f < \infty\} \cup \{x \in \mathbb{R} : 0 \in B, f(x) = \pm\infty\} \in \mathcal{M}$.

■

\hookrightarrow **Definition 2.3:** If a statement is true for every $x \in A$ where $A \in \mathcal{M}$ s.t. $m(A^c) = 0$, then we say the statement is true a.e. (almost everywhere).

\hookrightarrow **Proposition 2.6:** If $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is measurable and $f = g$ a.e. then g is measurable.

\hookrightarrow **Corollary 2.2:** If f is finite-valued a.e., then f is measurable $\Leftrightarrow f_{\mathbb{R}}$ is measurable $\Leftrightarrow \forall a < b \in \mathbb{R}, f^{-1}((a, b)) \in \mathcal{M}$.

\hookrightarrow **Proposition 2.7:** If $f \equiv c$ then f measurable.

If $f = \mathbb{1}_A$ for some $A \subseteq \mathbb{R}$, then f is measurable $\Leftrightarrow A \in \mathcal{M}$.

PROOF. Assume $f \equiv c$. Then

$$f^{-1}([-\infty, a)) = \begin{cases} \mathbb{R} & \text{if } c < a \\ \emptyset & \text{if } c \geq a \end{cases} \in \mathcal{M}.$$

Assume now $f = \mathbb{1}_A$. For all $a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) = \begin{cases} \mathbb{R} & \text{if } a > 1 \\ A^c & \text{if } 0 < a \leq 1 \in \mathcal{M} \Leftrightarrow A \in \mathcal{M}. \\ \emptyset & \text{if } a \leq 0 \end{cases}$$

■

\hookrightarrow **Proposition 2.8:** If f is (finite-valued) continuous, then f is measurable.

PROOF. $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous \Leftrightarrow for all $G \subseteq \mathbb{R}$ open, $f^{-1}(G)$ open. For all $a < b \in \mathbb{R}$, then $f^{-1}((a, b))$ open so $f^{-1}((a, b)) \in \mathcal{M}$ so f measurable.

In fact, if $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous, then for all $B \in \mathfrak{B}_{\mathbb{R}}, f^{-1}(B) \in \mathfrak{B}_{\mathbb{R}}$;

$$f^{-1}(\mathfrak{B}_{\mathbb{R}}) = f^{-1}(\sigma(\{\text{open sets}\})) = \sigma\left(\underbrace{f^{-1}(\{\text{open sets}\})}_{\text{all open}}\right) \subseteq \sigma(\{\text{open sets}\}) = \mathfrak{B}_{\mathbb{R}}.$$

Moreover, if f^{-1} (inverse) exists and is continuous, then for any $B \in \mathfrak{B}_{\mathbb{R}}, f(B) \in \mathfrak{B}_{\mathbb{R}}$. ■

↪ **Proposition 2.9:** If $f : \mathbb{R} \rightarrow \mathbb{R}$ is measurable and $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous, then $g \circ f$ is measurable.

Remark 2.1: The order matters! The converse doesn't hold in general.

PROOF. For all $a \in \mathbb{R}$,

$$\begin{aligned} (g \circ f)^{-1}((-\infty, a)) &= \{x \in \mathbb{R} : g(f(x)) < a\} \\ &= \{x \in \mathbb{R} : f(x) \in g^{-1}([-\infty, a))\} \\ &= f^{-1}(g^{-1}([-\infty, a))) \in \mathcal{M}. \end{aligned}$$

■

↪ **Proposition 2.10:** If $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$ is measurable, then:

1. for every $c \in \mathbb{R}$, cf is measurable (in particular $-f$ measurable);
2. $|f|$ is measurable;
3. for every $k \in \mathbb{N}$, f^k is a measurable.

PROOF. We prove just 3. If $k = 0$ this is trivial. For any $a \in \mathbb{R}$,

$$(f^k)^{-1}([-\infty, a)) = \begin{cases} f^{-1}([-\infty, a^{\frac{1}{k}})) & \text{if } k \text{ is odd} \\ \emptyset & \text{if } k \text{ is even and } a \leq 0 \in \mathcal{M}. \\ f^{-1}([-a^{\frac{1}{k}}, a^{\frac{1}{k}})) & \text{if } k \text{ is even and } a > 0 \end{cases}$$

■

↪ **Proposition 2.11:** If f, g are two finite-valued measurable functions, then $f + g, f \cdot g, f \vee g := \max\{f, g\}, f \wedge g := \min\{f, g\}$ are measurable functions, where

$$(f \vee g)(x) = \max\{f(x), g(x)\}.$$

PROOF. For all $a \in \mathbb{R}$,

$$\begin{aligned} (f + g)^{-1}([-\infty, a)) &= \{x \in \mathbb{R} : f(x) + g(x) < a\} \\ &= \{x \in \mathbb{R} : f(x) < a - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \{x \in \mathbb{R} : f(x) < q < a - g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \underbrace{\{x \in \mathbb{R} : f(x) < q\}}_{\in \mathcal{M}} \cap \underbrace{\{x \in \mathbb{R} : g(x) < a - q\}}_{\in \mathcal{M}} \in \mathcal{M}. \end{aligned}$$

This implies, then, that $f - g$ measurable, as are $(f + g)^2$ and $(f - g)^2$, and thus

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$$

is measurable.

We have too that

$$f \vee g = \frac{1}{2}(|f-g| + (f+g))$$

and so is measurable, and so

$$f \wedge g = -\max\{-f, -g\} = -(-f \vee -g)$$

is measurable. ■

↪ **Corollary 2.3:** If f is measurable, then $f^+ := f \vee 0 = \max\{f, 0\}$ and $f^- := -(f \wedge 0) = \max\{-f, 0\}$ are measurable, as is $f \wedge k$ for any $k \in \mathbb{R}$.

Remark 2.2: Notice that $f = f^+ - f^-$, even with “infinities”, and $|f| = f^+ + f^-$.

↪ **Proposition 2.12:** Let $\{f_n\}$ be a sequence of measurable functions. Then, $\sup_n f_n$, $\inf_n f_n$, $\limsup_{n \rightarrow \infty} f_n$, and $\liminf_{n \rightarrow \infty} f_n$ are all measurable (where $(\limsup_{n \rightarrow \infty} f_n)(x) := \limsup_{n \rightarrow \infty} f_n(x) = \inf_{m \geq 1} \sup_{n \geq m} f_n(x) = \lim_{m \rightarrow \infty} \sup_{n \geq m} f_n(x)$).

PROOF. To show $\sup_n f_n$ measurable, we will show for all $a \in \mathbb{R}$ $\{\sup_n f_n \leq a\} \in \mathcal{M}$.

$$x \in \left\{ \sup_n f_n \leq a \right\} \Leftrightarrow \sup_n f_n(x) \leq a \Leftrightarrow f_n(x) \leq a \forall n \geq 1 \Leftrightarrow x \in \bigcap_{n=1}^{\infty} \{f_n \leq a\},$$

hence $\{\sup_n f_n \leq a\} = \bigcap_{n=1}^{\infty} \underbrace{\{f_n \leq a\}}_{\in \mathcal{M}} \in \mathcal{M}$ and hence $\sup_n f_n$ is measurable. Note that

using \leq was important; $\{\sup_n f_n < a\} \not\subseteq \bigcap_{n=1}^{\infty} \{f_n < a\}$, since the $\sup_n f_n$ could equal a . We could say the following, however:

$$\left\{ \sup_n f_n < a \right\} = \bigcup_{k=1}^{\infty} \left\{ \sup_n f_n \leq a - \frac{1}{k} \right\} = \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{ f_n \leq a - \frac{1}{k} \right\} \in \mathcal{M}.$$

Next, we have $\inf_n f_n = -\sup_n(-f_n)$ so we are done.

For \limsup , \liminf , we have

$$\limsup_n f_n = \inf_{m \geq 1} \underbrace{\sup_{n \geq m} f_n}_{:= g_m}.$$

g_m is measurable for each $m \geq 1$, hence $\inf_m g_m$ is measurable, hence $\limsup_n f_n$ is measurable. Similar logic follows for \liminf .

We could have show, more directly, that

$$\begin{aligned}
\left\{ \limsup_n f_n < a \right\} &= \left\{ \inf_{m \geq 1} \sup_{n \geq m} f_n < a \right\} \\
&= \bigcup_{m=1}^{\infty} \left\{ \sup_{n \geq m} f_n < a \right\} \\
&= \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \left\{ \sup_{n \geq m} f_n \leq a - \frac{1}{k} \right\} \\
&= \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ f_n \leq a - \frac{1}{k} \right\}.
\end{aligned}$$

■

↪ **Proposition 2.13:** Let $\{f_n\}$ be a sequence of measurable functions. Then, all of the following sets are also measurable:

$$\begin{aligned}
\{x \in \mathbb{R} : \lim_{n \rightarrow \infty} f_n(x) \text{ exists in } \mathbb{R}\} &=: \left\{ \lim_{n \rightarrow \infty} f_n \text{ exists in } \mathbb{R} \right\}, \\
\{\lim f_n = \infty\}, \{\lim f_n = -\infty\}, \{\lim f_n = c \in \mathbb{R}\}.
\end{aligned}$$

Moreover, if $\lim_{n \rightarrow \infty} f_n$ exists (in \mathbb{R} or as $\pm\infty$) a.e. with $f = \lim_{n \rightarrow \infty} f_n$ a.e. then f is measurable.

PROOF. We have

$$\begin{aligned}
\{\lim f_n \text{ exists in } \mathbb{R}\} &= \{\limsup f_n = \liminf f_n \text{ and } -\infty < \limsup f_n < \infty\} \\
&= \{-\infty < \liminf f_n < \infty\} \cap \{-\infty < \limsup f_n < \infty\} \cap \{\limsup f_n - \liminf f_n = 0\} \in \mathcal{M}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\{\lim f_n = c\} &= \left\{ x \in \mathbb{R} : \forall k \geq 1, \exists n \geq 1 \text{ s.t. } \forall m \geq n, |f_n(x) - c| \leq \frac{1}{k} \right\} \\
&= \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \left\{ |f_n(x) - c| \leq \frac{1}{k} \right\}. \\
&\quad \forall \varepsilon = \frac{1}{k} > 0 \quad \exists n \geq 1 \quad \forall m \geq n
\end{aligned}$$

■

§2.2 Approximation by Simple Functions

Given a function $f : \mathbb{R} \rightarrow \overline{\mathbb{R}}$, measurable, we may write

$$f = f^+ - f^-,$$

where f^+, f^- are non-negative measurable functions; so, it suffices to study non-negative measurable functions. For any $n \geq 1$, we have

$$f_n^+ := (f^+ \wedge n) \cdot \mathbb{1}_{[-n, n]},$$

i.e., we cap f^+ at n , and disregard values of f^+ outside of $[-n, n]$; hence we limit our view to a $2n \times n$ “box”. Then, f_n^+ is non-negative, measurable, bounded (by n), compactly supported (zero outside a bounded set), and in particular $f_n^+ \uparrow$, with limit

$$\lim_{n \rightarrow \infty} f_n^+ = f^+.$$

An identical construction follows for f^- with

$$f_n^- := (f^- \wedge n) \mathbb{1}_{[-n, n]},$$

with $f_n^- \uparrow$ and

$$\lim_{n \rightarrow \infty} f_n^- = f^-.$$

Fix some n and consider f_n^+ . For $k = 0, 1, 2, \dots, 2^n n$, define

$$A_{n,k} := \left\{ x \in [-n, n] : \frac{k}{2^n} \leq f_n^+(x) < \frac{k+1}{2^n} \right\} = \left\{ \frac{k}{2^n} \leq f_n^+ < \frac{k+1}{2^n} \right\} \cap [-n, n] \in \mathcal{M},$$

noting that $A_{n,k} \cap A_{n,\ell} = \emptyset$ if $k \neq \ell$. Set now

$$\varphi_n := \sum_{k=0}^{n \cdot 2^n} \mathbb{1}_{A_{n,k}} \frac{k}{2^n} = \sum_{k=0}^{n \cdot 2^n} \begin{cases} \frac{k}{2^n} & \text{if in } A_{n,k}. \\ 0 & \text{else} \end{cases}.$$

We call φ_n a “simple function”; more generally:

↪ **Definition 2.4:** φ is a *simple function* if $\varphi = \sum_{k=1}^L \mathbb{1}_{E_k} \cdot a_k$ where L a positive integer, a_k 's are constant, E_k 's are measurable sets of finite measure.

Moreover, note that $\varphi_n \uparrow$; at each new stage $n \rightarrow n+1$, the regions are cut in two, $A_{n,k} = A_{n+1,2k} \cup A_{n+1,2k+1}$. In addition, we have $\varphi_n \leq f_n^+ \leq f^+$ for all n . Moreover, we have the following:

↪ **Proposition 2.14:**

$$\lim_{n \rightarrow \infty} \varphi_n(x) = f^+(x)$$

for all $x \in \mathbb{R}$.

PROOF. For all $x \in \mathbb{R}$, for sufficiently large n we have that $x \in [-n, n]$ and so $f^+(x) = f^+(x) \mathbb{1}_{[-n, n]}(x)$. Assume for now $f^+ < \infty$. Then, for sufficiently large (r?) n , we can ensure $f^+(x) < n$ and so $f^+(x) = f_n^+(x)$ for such an x . Further, we have that $x \in A_{n,k}$ for some k so $\varphi_n(x) = \frac{k}{2^n}$ and $f_n^+(x) < \frac{k+1}{2^n}$ and thus

$$0 \leq f_n^+(x) - \varphi_n(x) < \frac{k+1}{2^n} - \frac{k}{2^n} = 2^{-n}$$

by construction and so $0 \leq f^+(x) - \varphi_n(x) \leq 2^{-n}$ and thus $\lim_{n \rightarrow \infty} \varphi_n(x) = f^+(x)$.

In the case that $f^+(x) = \infty$, then $\varphi_n(x) = n$ for all sufficiently large n hence

$$\lim_{n \rightarrow \infty} \varphi_n(x) = \lim_{n \rightarrow \infty} n = \infty = f^+(x).$$

■

↪ **Theorem 2.1:** If g is measurable and non-negative, there exists a sequence of simple functions $\{\varphi_n\}$ such that $\varphi_n \uparrow$ and $\lim_{n \rightarrow \infty} \varphi_n(x) = g(x)$ for every $x \in \mathbb{R}$.

We can repeat this same construction and proof for f^- with a sequence $\{\tilde{\varphi}_n\}$. Even better:

↪ **Theorem 2.2:** If f is measurable, then \exists a sequence of simple functions $\{\psi_n\}$ such that $|\psi_n| \uparrow$ and $|\psi_n| \leq |f|$ for all n and for all $x \in \mathbb{R}$, $\lim_{n \rightarrow \infty} \psi_n(x) = f(x)$.

PROOF. Take $\psi_n = \varphi_n - \tilde{\varphi}_n$ as above; then for all $x \in \mathbb{R}$, at least one of $\varphi_n(x), \tilde{\varphi}_n(x)$ equals zero. Then

$$|\psi_n| = \varphi_n + \tilde{\varphi}_n < f^+ + f^- = |f|,$$

and

$$\lim_{n \rightarrow \infty} \psi_n(x) = \lim_{n \rightarrow \infty} \varphi_n(x) - \lim_{n \rightarrow \infty} \tilde{\varphi}_n(x) = f^+ - f^- = f.$$

■

↪ **Definition 2.5 (Step Function):** θ a *step function* if it takes the form

$$\theta(x) = \sum_{k=1}^L a_k \mathbb{1}_{I_k}(x),$$

where $L \in \mathbb{N}, a_k$'s constant, and I_k finite, open intervals.

↪ **Theorem 2.3:** If f is measurable, then there exists a sequence of step functions $\{\theta_n\}$ such that

$$\lim_{n \rightarrow \infty} \theta_n(x) = f(x) \text{ for **almost every** } x \in \mathbb{R}.$$

In particular, we do not have pointwise convergence as for general simple functions, but we have convergence outside a zero-measure set.

PROOF. Assume, wlog, that f non-negative (by the previous construction, we can "split" f if not and approximate its positive, negative parts). Given $A \in \mathcal{M}$ with finite measure, recall that for every $\varepsilon > 0$, there exists finitely many finite open intervals I_1, \dots, I_N such that

$$m\left(A \Delta \left(\bigcup_{i=1}^N I_i\right)\right) < \varepsilon.$$

By renaming/rearranging I_i 's if necessary, we may assume that I_i 's are disjoint; hence

$$\mathbb{1}_{\bigcup_{i=1}^N I_i} = \sum_{i=1}^N \mathbb{1}_{I_i}.$$

Put

$$\theta_A := \sum_{i=1}^N \mathbb{1}_{I_i},$$

noting this is indeed a step function as the name suggests. Then, remark that

$$m(\underbrace{\{x \in \mathbb{R} : \mathbb{1}_A(x) \neq \theta_A(x)\}}_{=A \Delta (\bigcup_{i=1}^N I_i)}) < \varepsilon.$$

Since f measurable and non-negative, $\exists \{\varphi_n\}$ sequence of simple functions with limit f . In particular,

$$\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}}.$$

Applying our above analysis to each $A_{n,k}$, then, we have that for any $n \geq 1$ and $k = 0, 1, \dots, n2^n$ we can find a step function $\theta_{n,k}$ such that

$$m(\{x \in \mathbb{R} : \mathbb{1}_{A_{n,k}} \neq \theta_{n,k}(x)\}) < \frac{1}{2^n(n2^n + 1)} \quad (" = \varepsilon").$$

Put then

$$\theta_n := \sum_{k=0}^{n2^n} \frac{k}{2^n} \theta_{n,k},$$

which is itself a step function. Put

$$E_n := \{x \in \mathbb{R} : \theta_n(x) \neq \varphi_n(x)\}.$$

Then,

$$m(E_n) \leq m\left(\bigcup_{k=0}^{n2^n} \{\theta_{n,k} \neq \mathbb{1}_{A_{n,k}}\}\right) \leq \sum_{k=0}^{n2^n} m(\{\theta_{n,k} \neq \mathbb{1}_{A_{n,k}}\}) \leq 2^{-n}.$$

The φ_n 's are chosen such that $\forall x \in \mathbb{R}, |\varphi_n(x) - f_n(x)| \leq \frac{1}{2^n}$. Putting

$$F_n := \{x \in \mathbb{R} : |\theta_n(x) - f_n(x)| > 2^{-n}\},$$

then remark that $F_n \subseteq E_n$ so $m(F_n) \leq \frac{1}{2^n}$.

We claim now that for a.e. $x \in \mathbb{R}$, $\exists m \geq 1$ such that $\forall n \geq m, |\theta_n(x) - f_n(x)| \leq \frac{1}{2^n}$, remarking that such an m is *dependent* on x . Consider the complement of this statement; if this set has measure 0, we are done. The logical negation would be "for every $m \geq 1$, exist $n \geq m$ such that $|\theta_n(x) - f_n(x)| > 2^{-n}$ ", which is equivalent to the set

$$\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in \mathbb{R} : |\theta_n(x) - f_n(x)| > 2^{-n}\} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n.$$

Let $B_m := \bigcup_{n=m}^{\infty} F_n$; notice $B_m \downarrow$. Then, by continuity from above ****

$$m\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n\right) = \lim_{m \rightarrow \infty} m(B_m) \leq \lim_{m \rightarrow \infty} \sum_{n=m}^{\infty} m(F_n) \leq \lim_{m \rightarrow \infty} \sum_{n=m}^{\infty} \frac{1}{2^n} = 0,$$

since the tail of a convergent series must converge to zero. Hence, the set has measure 0 as desired so for almost every $x \in \mathbb{R}$ there exists $m \geq 1$ such that for all $n \geq m$, $|\theta_n - f_n| \leq \frac{1}{2^n}$, hence almost everywhere $\lim_{n \rightarrow \infty} (\theta_n - f_n) = 0$. Therefore, almost everywhere,

$$\theta_n = (\theta_n - f_n) + f_n \xrightarrow{n \rightarrow \infty} f.$$

■

In this proof, we have proven (and then used) more generally:

↪ **Lemma 2.1** (Borel-Cantelli Lemma): If $\{F_n\} \subseteq \mathcal{M}$ such that $\sum_{n=1}^{\infty} m(F_n) < \infty$, then

$$m\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n\right) = 0.$$

PROOF. Remark that $\bigcup_{n=m}^{\infty} F_n$ a decreasing sequence of functions indexed by m . By continuity of the measure and subadditivity,

$$m\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} F_n\right) = \lim_{m \rightarrow \infty} m\left(\bigcup_{n=m}^{\infty} F_n\right) \leq \lim_{m \rightarrow \infty} \sum_{n=m}^{\infty} m(F_n) = 0,$$

since the tail of a converging sequence must converge to zero. ■

§2.3 Convergence Almost Everywhere vs Convergence in Measure

↪ **Definition 2.6** (Convergence Almost Everywhere): For measurable functions $\{f_n\}, f$ we say f_n converges to f a.e. and write $f_n \rightarrow f$ a.e. if for almost every $x \in \mathbb{R}$, $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

Similarly, we say $f_n \rightarrow f$ a.e. on A if $\exists B \subseteq A$ with $m(B) = 0$ such that $\forall x \in A - B$, $\lim_{n \rightarrow \infty} f_n(x) = f(x)$.

↪ **Definition 2.7** (Convergence in Measure): For measurable, finite-valued functions $\{f_n\}, f$ we say f_n converges to f in measure and write $f_n \rightarrow f$ in measure if for every $\delta > 0$,

$$\lim_{n \rightarrow \infty} m(\{x \in \mathbb{R} : |f_n(x) - f(x)| \geq \delta\}) = 0.$$

Similarly, we say $f_n \rightarrow f$ in measure on A if $\forall \delta > 0$, $\lim_{n \rightarrow \infty} m(\{x \in A : |f_n(x) - f(x)| \geq \delta\}) = 0$.

↪ **Proposition 2.15**: Given finite-valued measurable functions $\{f_n\}, f$ and $A \in \mathcal{M}$ with finite measure, then if $f_n \rightarrow f$ a.e. on A , then $f_n \rightarrow f$ in measure on A .

PROOF. For all $\delta > 0$,

$$\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in A : |f_n(x) - f(x)| > \delta\} \subseteq \{x \in A : \lim_{n \rightarrow \infty} f_n(x) \neq f(x)\}.$$

The set on the RHS has measure zero and thus so does the left one. Then,

$$\lim_{m \rightarrow \infty} m\left(\bigcup_{n=m}^{\infty} \{x \in A : |f_n(x) - f(x)| > \delta\}\right) = 0$$

by continuity, and

$$\{|f_m - f| > \delta\} \subseteq \bigcup_{n=m}^{\infty} \{|f_n - f| > \delta\}$$

hence $m(\{|f_m - f| > \delta\}) \leq m(\bigcup_{n=m}^{\infty} \{|f_n - f| > \delta\}) \xrightarrow{m \rightarrow \infty} 0$. ■

⊗ **Example 2.1:** We give an example of why the assumption that $m(A) < \infty$ is necessary. Let, $f_n = \mathbb{1}_{[n, \infty)}$ and $f \equiv 0$. Then, $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ for every $x \in \mathbb{R}$. But $m(\{x \in \mathbb{R} : |f_n(x) - f(x)| = 1\}) = m([n, \infty)) = \infty$.

In general, the converse statement $f_n \rightarrow f$ in measure does *not* imply that $f_n \rightarrow f$ almost everywhere, even on finite measure sets. Put $\varphi_{1,1} = \mathbb{1}_{[0,1)}$, $\varphi_{2,1} = \mathbb{1}_{[0, \frac{1}{2})}$, $\varphi_{2,2} = \mathbb{1}_{[\frac{1}{2}, 1)}$, $\varphi_{3,1} = \mathbb{1}_{[0, \frac{1}{3})}$, $\varphi_{3,2} = \mathbb{1}_{[\frac{1}{3}, \frac{2}{3})}$, $\varphi_{3,3} = \mathbb{1}_{[\frac{2}{3}, 1)}$, or in general $\varphi_{k,j} = \mathbb{1}_{[\frac{j-1}{k}, \frac{j}{k})}$ for $j = 1, \dots, k$. Reorder $\varphi_{k,j}$ “lexicographically” into $\{f_n\}$. Then, we claim $f_n \rightarrow 0$ in measure on $[0, 1)$; for any $\delta \in (0, 1)$,

$$m(\{|f_n - 0| > \delta\}) = \frac{1}{k(n)} \rightarrow 0,$$

where $k(n)$ the “row” that f_n comes from. Hence, f_n converges in measure. However, f_n does not converge almost everywhere on $[0, 1)$. Indeed, for each $x \in \mathbb{R}$ and $k \geq 1$, there exists a *unique* j such that $x \in [\frac{j-1}{k}, \frac{j}{k})$ hence $\varphi_{k,j}(x) = 1$, so in other notation there always exists an n such that $f_n(x) = 1$, and so precisely $f_n(x) = 1$ for infinitely many n . Hence, we do not have convergence everywhere (in fact, anywhere).

↔ **Proposition 2.16:** Given $\{f_n\}, f$ measurable, finite-valued functions, if $f_n \rightarrow f$ in measure, then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \rightarrow f$ a.e. as $k \rightarrow \infty$.

PROOF. Assume $f_n \rightarrow f$ in measure, that is for every $\delta > 0$, $m(\{|f_n - f| > \delta\}) \rightarrow 0$.

Hence, for all $k \geq 1$, with $\delta = \frac{1}{k}$, we have that for some sufficiently large n_k , we have

that $m\left(\underbrace{\{|f_{n_k} - f| > \frac{1}{k}\}}_{:=A_k}\right) \leq \frac{1}{k^2}$, hence $\sum_{k=1}^{\infty} m(A_k) < \infty$. Hence,

$$m\left(\bigcap_{\ell=1}^{\infty} \bigcup_{k=\ell}^{\infty} A_k\right) = \lim_{\ell \rightarrow \infty} m\left(\bigcup_{k=\ell}^{\infty} A_k\right) \leq \lim_{\ell \rightarrow \infty} \sum_{k=\ell}^{\infty} m(A_k) = 0,$$

since $\sum_{k=\ell}^{\infty} m(A_k)$ the tail of a converging series. Hence, complementing the above, a.e. there $\exists \ell$ such that for every $k \geq \ell$, $|f_{n_k} - f| \leq \frac{1}{k}$ and so $\lim_{k \rightarrow \infty} |f_{n_k} - f| = 0$ almost everywhere, and so $f_{n_k} \rightarrow f$ a.e. (as $k \rightarrow \infty$). ■

↔ **Proposition 2.17** (Subsequence Test): Given $\{f_n\}$, f measurable, finite-valued functions, $f_n \rightarrow f$ in measure \Leftrightarrow for every subsequence $\{n_k\}$, there exists a subsubsequence $\{n_{k_\ell}\} \subset \{n_k\}$ such that $f_{n_{k_\ell}} \rightarrow f$ in measure as $\ell \rightarrow \infty$.

PROOF. \Rightarrow is clear. For \Leftarrow , suppose towards a contradiction that $f_n \not\rightarrow f$ in measure. Then, $\exists \delta > 0$ and subsequence $\{n_k\}$ $m(\{|f_{n_k} - f| > \delta\}) > \delta$ for every k . By the assumption of the RHS, there exists a further subsequence $\{n_{k_\ell}\}$ such that $f_{n_{k_\ell}} \rightarrow f$ in measure. This is a contradiction. ■

⊗ **Example 2.2** (Assignment Exercise): Prove that if $f_n \rightarrow f$ in measure and $g_n \rightarrow g$ in measure, $f_n g_n \rightarrow f g$ in measure (everything finite valued, measurable).

§2.4 Egorov's Theorem and Lusin's Theorem

Recall that if f is measurable, then $\exists \{\theta_n\}$ sequence of step functions such that $\theta_n \rightarrow f$ almost everywhere.

↔ **Theorem 2.4** (Egorov's): Given $\{f_n\}$, f measurable functions and $A \in \mathcal{M}$ with $m(A) < \infty$, if $f_n \rightarrow f$ a.e. on A , then $\forall \varepsilon > 0$, there exists a closed subset $A_\varepsilon \subseteq A$ with $m(A \setminus A_\varepsilon) \leq \varepsilon$ such that $f_n \rightarrow f$ uniformly on A_ε .

PROOF. We assume first f is finite-valued on A (otherwise, replace A with $A \cap \{-\infty < f < \infty\}$; we'll deal with $\{f = \pm\infty\}$ later). We want to show that $\forall \varepsilon > 0, \exists$ closed $A_\varepsilon \subseteq A$ s.t. $m(A \setminus A_\varepsilon) < \varepsilon$ and $\sup_{x \in A_\varepsilon} |f_n(x) - f(x)| \rightarrow 0$ as $n \rightarrow \infty$.

For each $k \geq 1$ and $n \geq 1$, put

$$E_n^{(k)} := \left\{ x \in A : |f_j(x) - f(x)| \leq \frac{1}{k} \forall j \geq n \right\}.$$

For fixed k , remark that $E_n^{(k)} \subseteq E_{n+1}^{(k)}$, i.e. $E_n^{(k)}$ increasing (wrt n), so we may consider

$$\bigcup_{n=1}^{\infty} E_n^{(k)} = \left\{ x \in A : \exists n \geq 1 \text{ s.t. } \forall j \geq n, |f_j(x) - f(x)| \leq \frac{1}{k} \right\} \supseteq \left\{ x \in A : \lim_{n \rightarrow \infty} f_n(x) = f(x) \right\} =: A'.$$

By assumption, $m(A') = m(A)$, so by continuity and the superset relation above, $m(A) = m(A') \leq m\left(\bigcup_{n=1}^{\infty} E_n^{(k)}\right) = \lim_{n \rightarrow \infty} m\left(E_n^{(k)}\right) \leq m(A)$, and thus $\lim_{n \rightarrow \infty} m\left(E_n^{(k)}\right) = m(A)$ for every $k \geq 1$.

Given, then, any $\varepsilon > 0$, there exists a n_k such that $m\left(A \setminus E_{n_k}^{(k)}\right) = m(A) - m\left(E_{n_k}^{(k)}\right) < \frac{1}{2^k} \frac{\varepsilon}{2}$. Set

$$B := A \setminus \left(\bigcap_{k=1}^{\infty} E_{n_k}^{(k)} \right),$$

then

$$m(B) = m\left(\bigcup_{k=1}^{\infty} A \setminus E_{n_k}^{(k)} \right) \leq \sum_{k=1}^{\infty} m(A \setminus E_{n_k}^{(k)}) \leq \frac{\varepsilon}{2}.$$

Put

$$\tilde{A} := A \setminus B = \bigcap_{k=1}^{\infty} E_{n_k}^{(k)}.$$

Then, if $x \in \tilde{A}$, then $x \in E_{n_k}^{(k)}$ for every k , and hence for every $k \geq 1$ and $j \geq n_k$, $|f_j(x) - f(x)| \leq \frac{1}{k}$. This shows then that $f_n \rightarrow f$ uniformly on \tilde{A} . By regularity of m , there exists a closed $A_\varepsilon \subseteq \tilde{A}$ such that $m(\tilde{A} \setminus A_\varepsilon) \leq \frac{\varepsilon}{2}$. Then, $f_n \rightarrow f$ uniformly on A_ε , and $m(A \setminus A_\varepsilon) = m(A \setminus \tilde{A}) + m(\tilde{A} \setminus A_\varepsilon) < \varepsilon$.

Now, if $f = \infty / -\infty$ on A , then $A = A^\infty \cup A^{-\infty} \cup A^\mathbb{R}$ (with $A^\bullet := \{f = \bullet\} \cap A$). The last case is done. For A^∞ (similar construction for $A^{-\infty}$), define for every $k, n \geq 1$,

$$E_n^{(k)} := \{x \in A : f_j(x) > k \forall j \geq n\}.$$

Then, the remainder of the proof follows precisely the same for the sequence of sets $E_n^{(k)}$. ■

Remark 2.3:

1. The assumption $m(A) < \infty$ is necessary. For instance $f_n = \mathbb{1}_{[n, \infty)} \rightarrow 0$ pointwise, but for any $a \in \mathbb{R}$, f_n does not converge to 0 uniformly on (a, ∞) .
2. In general, Egorov's $\nRightarrow f_n \rightarrow f$ uniformly a.e.. For instance, on $[0, 1]$, let $f_n(x) = x^n$ and $f(x) \equiv 0$. For every $x \in [0, 1)$, $f_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$. Hence, $f_n \rightarrow f$ a.e. on $[0, 1]$ (the only point that doesn't converge, indeed, is at 1). If $A \subseteq [0, 1]$ is closed such that $1 \in A$, then $f_n \nrightarrow f$ uniformly on A . To see this, let $\{x_m\} \subseteq A$ such that $x_m \uparrow$ and $\lim_{m \rightarrow \infty} x_m = 1$. Then, for any fixed n ,

$$\sup_{x \in A} |f_n(x) - f(x)| \geq \sup_m |f_n(x_m) - f(x_m)| = \sup_m x_m^n = 1,$$

hence f_n does not converge uniformly on A .

↔ Theorem 2.5 (Lusin's Theorem): Given f measurable and finite-valued and $A \in \mathcal{M}$ with $m(A) < \infty$, for all $\varepsilon > 0$, there exists a closed $A_\varepsilon \subseteq A$ with $m(A \setminus A_\varepsilon) < \varepsilon$ such that $f|_{A_\varepsilon}$ is continuous.

Remark 2.4: Lusin's Theorem states that $f|_{A_\varepsilon}$ is continuous as a function on A_ε , which is *not* the same as saying f as a function on A is continuous at points in A_ε .

For instance, $f = \mathbb{1}_{\mathbb{Q} \cap [0,1]}$ is not continuous anywhere on $[0, 1]$. However, $f|_{\mathbb{Q} \cap [0,1]}$ is constant and therefore continuous on $\mathbb{Q} \cap [0, 1]$.

PROOF. Let $\{\theta_n\}$ be a sequence of step functions such that $\theta_n \rightarrow f$ a.e. on A . Note that θ_n piecewise constant and hence piecewise continuous. Given $\varepsilon > 0$ and $n \geq 1$, we can find an open set E_n such that $\theta_n|_{E_n^c}$ is continuous and $m(E_n) \leq \frac{\varepsilon}{2^{2^n}}$. Meanwhile, Egorov's implies that there exists a closed $B \subseteq A$ such that $m(A \setminus B) \leq \frac{\varepsilon}{2}$ such that $\theta_n \rightarrow f$ uniformly on B . Set

$$A_\varepsilon = B \setminus \bigcup_{n=1}^{\infty} E_n,$$

noting that $A_\varepsilon \subset A$ closed and

$$m(A \setminus A_\varepsilon) = m(A \setminus B) + m\left(\bigcup_{n=1}^{\infty} E_n\right) = \frac{\varepsilon}{2} + \sum_{n=1}^{\infty} m(E_n) \leq \varepsilon.$$

Finally, on A_ε , $\theta_n \rightarrow f$ uniformly and $\theta_n|_{A_\varepsilon}$ continuous, and hence $f|_{A_\varepsilon}$ continuous (uniform limit of continuous functions is continuous). ■

Remark 2.5:

1. Lusin's Theorem $\nRightarrow f$ is continuous almost everywhere in general. For instance, recall that fat Cantor set \tilde{C} , with $m(\tilde{C}) = \frac{1}{2}$. Let $f = \mathbb{1}_{\tilde{C}}$. f is NOT continuous a.e. on $[0, 1]$, i.e. $\forall B \subseteq [0, 1]$ with $m(B) = 1$, $f|_B$ is NOT continuous. To see this, let $\tilde{D} = [0, 1] \setminus \tilde{C}$. Since $m(B) = 1$, then $m(\tilde{C} \cap B) = m(\tilde{D} \cap B) = \frac{1}{2}$. Then for any $x \in \tilde{C} \cap B$, $f|_B$ is NOT continuous at x . If it were at say some $x_0 \in \tilde{C} \cap B$, then there must exist some $\delta > 0$ such that for any $x \in (x_0 - \delta, x_0 + \delta) \cap B$, $|f(x) - f(x_0)| < \frac{1}{2}$. Hence, for any $x \in (x_0 - \delta, x_0 + \delta) \cap B$, $\frac{1}{2} \leq f(x) \leq \frac{3}{2}$. However, $m((x_0 - \delta, x_0 + \delta) \cap B \cap \tilde{D}) > 0$ so it must be that $\exists y \in (x_0 - \delta, x_0 + \delta) \cap B \cap \tilde{D} \Rightarrow f(y) = 0$, a contradiction. How, then, does one apply Lusin's; that is, $\forall \varepsilon > 0$, there must exist some $A_\varepsilon \subseteq [0, 1]$ such that $m([0, 1] \setminus A_\varepsilon) < \varepsilon$ and $f|_{A_\varepsilon} < \varepsilon$ (exercise)?
2. (Exercise) The $\{\theta_n\}$'s are not continuous on \mathbb{R} , but you can choose a sequence $\{\tilde{\theta}_n\}$ to be continuous on \mathbb{R} such that $\tilde{\theta}_n \rightarrow f$ a.e..
3. Lusin's Theorem $\Rightarrow \forall k$ sufficiently large, $\exists A_k \subseteq A$ closed such that $m(A \setminus A_k) \leq \frac{1}{k}$ and $f|_{A_k}$ continuous on A_k . In fact, we can construct them such that $A_k \uparrow$ (otherwise replace A_k with $\bigcup_{i=1}^k A_i$).

§2.5 Construction of Integrals

2.5.1 Integral of Simple Functions

↪ **Definition 2.8:** Given a simple function $\varphi = \sum_{k=1}^L a_k \mathbb{1}_{E_k}$, the (Lebesgue) integral of φ is defined as

$$\int_{\mathbb{R}} \varphi(x) dx = \int_{\mathbb{R}} \varphi := \sum_{k=1}^L a_k \cdot m(E_k).$$

For any $A \in \mathcal{M}$, $\mathbb{1}_A \varphi$ is again a simple function and we define

$$\int_A \varphi := \int_{\mathbb{R}} \mathbb{1}_A \varphi.$$

↪ **Proposition 2.18** (Properties of $\int_{\mathbb{R}} \varphi$):

1. (Well-definedness) The written representation of φ is not necessarily unique, but if $\varphi = \sum_{k=1}^L a_k \mathbb{1}_{E_k} = \sum_{\ell=1}^M b_{\ell} \mathbb{1}_{F_{\ell}}$, then

$$\sum_{k=1}^L a_k m(E_k) = \sum_{\ell=1}^M b_{\ell} m(F_{\ell}).$$

2. (Linearity) If φ, ψ two simple functions and $a, b \in \mathbb{R}$, then $a\varphi + b\psi$ a simple function, and

$$\int_{\mathbb{R}} a\varphi + b\psi = a \cdot \int_{\mathbb{R}} \varphi + b \cdot \int_{\mathbb{R}} \psi.$$

3. (Finite Additivity) If φ a simple function, $A, B \in \mathcal{M}$ with $A \cap B = \emptyset$, then

$$\int_{A \cup B} \varphi = \int_A \varphi + \int_B \varphi.$$

4. (Monotonicity) If φ, ψ are two simple functions with $\varphi \leq \psi$, then $\int_{\mathbb{R}} \varphi \leq \int_{\mathbb{R}} \psi$.
5. If φ a simple function then so is $|\varphi|$ and $|\int_{\mathbb{R}} \varphi| \leq \int_{\mathbb{R}} |\varphi|$.

PROOF.

1. wlog, we may assume E_k and F_{ℓ} are respectively disjoint. Set $a_0 = b_0 = 0$, $E_0 := \left(\bigcup_{k=1}^L E_k\right)^c$, $F_0 := \left(\bigcup_{\ell=1}^M F_{\ell}\right)^c$ for convenience. Now, $\{E_0, \dots, E_L\}, \{F_0, \dots, F_M\}$ are two partitions of \mathbb{R} . In particular, then, for each k , $\mathbb{1}_{E_k} = \sum_{\ell=0}^M \mathbb{1}_{E_k \cap F_{\ell}}$, since $E_k = \bigsqcup_{\ell=0}^M (E_k \cap F_{\ell})$. Now, we have

$$\varphi = \sum_{k=0}^L a_k \mathbb{1}_{E_k} = \sum_{k=0}^L \sum_{\ell=0}^M a_k \mathbb{1}_{E_k \cap F_{\ell}}.$$

Similarly partitioning, we have

$$\varphi = \sum_{\ell=0}^M b_{\ell} \mathbb{1}_{F_{\ell}} = \sum_{\ell=0}^M \sum_{k=0}^L b_{\ell} \mathbb{1}_{E_k \cap F_{\ell}}.$$

If $E_k \cap F_{\ell} \neq \emptyset$, then $a_k = b_{\ell}$, and thus on the one hand

$$\int_{\mathbb{R}} \varphi = \sum_{k=0}^L \sum_{\ell=0}^M a_k m(E_k \cap F_{\ell})$$

and on the other

$$\int_{\mathbb{R}} \varphi = \sum_{\ell=0}^M \sum_{k=0}^L b_{\ell} m(E_k \cap F_{\ell}),$$

(with summation convention $0 \cdot \infty = 0$). If $m(E_k \cap F_{\ell}) > 0$, then $E_k \cap F_{\ell} \neq \emptyset$ and so $a_k = b_{\ell}$ and so the two sums agree.

4. Assume $\varphi = \sum_{k=1}^L a_k \mathbb{1}_{E_k}$, $\psi = \sum_{\ell=1}^M b_{\ell} \mathbb{1}_{F_{\ell}}$. Repeat the partitioning/rewriting steps from part 1, then note that since $\varphi \leq \psi$, if $E_k \cap F_{\ell} \neq \emptyset$, it must be that $a_k \leq b_{\ell}$, so if $m(E_k \cap F_{\ell}) > 0$ $a_k \leq b_{\ell}$ and thus the monotonicity follows. ■

2.5.2 Integral of Non-Negative Functions

↪ **Definition 2.9:** If f a non-negative, measurable function then the integral of f is given by

$$\int_{\mathbb{R}} f(x) dx = \int_{\mathbb{R}} f := \sup \left\{ \int_{\mathbb{R}} \varphi : \varphi \text{ is simple and } \varphi \leq f \right\}.$$

↪ **Proposition 2.19:** The definition above agrees with that for simple functions that are also non-negative, namely this definition is consistent with the previous.

PROOF. Let φ be non-negative. Then $\varphi \leq \varphi$ certainly so the first definition $\int_{\mathbb{R}} \varphi \leq \sup \{\dots\}$. Conversely, it suffices to show that for any non-negative simple $\psi \leq \varphi$, $\int_{\mathbb{R}} \psi \leq \int_{\mathbb{R}} \varphi$, using the first definition. But this simply follows from monotonicity of \int , and we are done. ■

Remark 2.6: Given $f \geq 0$ and measurable, this definition implies that there exists a sequence $\{\varphi_n\}$ of simple functions such that $\varphi_n \leq f$ and $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n = \int_{\mathbb{R}} f$. We would like to show that, in some sense, the choice of $\{\varphi_n\}$ is arbitrary.

↪ **Theorem 2.6:** Suppose $f \geq 0$ and measurable. If $\{\varphi_n\}$ a sequence of simple functions such that $\varphi_n \uparrow$ and $\lim_{n \rightarrow \infty} \varphi_n = f$ pointwise, then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n = \int_{\mathbb{R}} f.$$

PROOF. Since $\varphi_n \leq f$ for all $n \geq 1$, then $\int_{\mathbb{R}} \varphi_n \leq \int_{\mathbb{R}} f$ and so $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n \leq \int_{\mathbb{R}} f$ (nothing the limit on the LHS necessarily always exists by monotonicity). On the other hand, it suffices to show that $\forall \psi \leq f$ simple, that $\int_{\mathbb{R}} \psi \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n$. Assume $\psi = \sum_{k=1}^L a_k \mathbb{1}_{E_k} = \sum_{k=0}^L a_k \mathbb{1}_{E_k}$ where $\{E_0, \dots, E_L\}$ forms a partition of \mathbb{R} . Since

$$\int_{\mathbb{R}} \psi = \sum_{k=0}^L a_k m(E_k)$$

and

$$\int_{\mathbb{R}} \varphi_n = \sum_{k=0}^L \int_{E_k} \varphi_n$$

by finite additivity. It suffices to show then that for each $k = 0, \dots, L$, $a_k m(E_k) \leq \lim_{n \rightarrow \infty} \int_{E_k} \varphi_n$.

First, if $a_k = 0$ or $m(E_k) = 0$, then we are done. Assume $a_k, m(E_k) > 0$. For each fixed k , $\lim_{n \rightarrow \infty} \varphi_n = f \geq \psi$ so for every $x \in E_k$, $\lim_{n \rightarrow \infty} \varphi_n(x) \geq \psi(x) = a_k$. For any $\varepsilon > 0$, put

$$C_n^\varepsilon := \{x \in E_k : \varphi_n(x) \geq (1 - \varepsilon)a_k\}.$$

Since $\varphi_n \leq \varphi_{n+1}$, $C_n^\varepsilon \uparrow$ wrt n . Then note

$$\bigcup_{n=1}^{\infty} C_n^\varepsilon = E_k.$$

Then,

$$\lim_{n \rightarrow \infty} \int_{E_k} \varphi_n = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \mathbb{1}_{E_k} \varphi_n \geq \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \mathbb{1}_{C_n^\varepsilon} \varphi_n \geq \lim_{n \rightarrow \infty} (1 - \varepsilon)a_k m(C_n^\varepsilon) = (1 - \varepsilon)a_k m(E_k),$$

where we use the fact that $\mathbb{1}_{E_k} \varphi_n \geq \mathbb{1}_{C_n^\varepsilon} \varphi_n \geq (1 - \varepsilon)a_k \mathbb{1}_{C_n^\varepsilon}$ and $\lim_{n \rightarrow \infty} m(C_n^\varepsilon) = m(\bigcup_{n=1}^{\infty} C_n^\varepsilon) = m(E_k)$. Since ε arbitrary, then

$$\lim_{n \rightarrow \infty} \int_{E_k} \varphi_n \geq a_k m(E_k),$$

and we are done. ■

↪ **Corollary 2.4:** For any $f \geq 0$ measurable, if $\forall n \geq 1, k = 0, 1, \dots, n2^n$ with $A_{n,k} := \left\{ \frac{k}{2^n} \leq f < \frac{k+1}{2^n} \right\}$, then

$$\int_{\mathbb{R}} f = \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} m(A_{n,k}).$$

PROOF. Let $\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}}$, then $\varphi_n \uparrow$ and $\varphi_n \rightarrow f$. ■

↪ **Proposition 2.20** (Properties of Integral of Non-Negative Functions):

1. (Well-definedness) If $f, g \geq 0$ measurable such that $f = g$ a.e., then $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$.
2. (Linearity) For any $f, g \geq 0$ measurable and $a, b \geq 0$, then $\int_{\mathbb{R}} (af + bg) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g$.
3. (Monotonicity) If $f, g \geq 0$ measurable and $f \leq g$ a.e., then $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$.
4. i. Let $f \geq 0$ measurable, then $\int_{\mathbb{R}} f = 0 \Leftrightarrow f \equiv 0$ a.e.
 ii. Let $f \geq 0$ measurable, $A \in \mathcal{M}$. Then $\int_A f = 0 \Leftrightarrow$ either $f \equiv 0$ a.e. on A or $m(A) = 0$.
 iii. Let $f \geq 0$ measurable, then if $\int_{\mathbb{R}} f < \infty$ then f is finite valued a.e.
5. (Markov's Inequality) Let $f \geq 0$ measurable and $0 < a < \infty$. Then, $m(\{f > a\}) \leq \frac{1}{a} \int_{\mathbb{R}} f$. In particular, if the RHS is finite, $\lim_{a \rightarrow \infty} m(\{f > a\}) = 0$, in fact in $O\left(\frac{1}{a}\right)$.

PROOF.

1. Let $\{\varphi_n\}, \{\psi_n\}$ sequences of simple functions such that both are monotonically increasing with $\varphi_n \rightarrow f, \psi_n \rightarrow g$. Put $h_n := \varphi_n \mathbb{1}_{\{f=g\}} + \psi_n \mathbb{1}_{\{f \neq g\}}$; then h_n again simple, $h_n \uparrow$, and $h_n \rightarrow g$ everywhere. Then,

$$\int_{\mathbb{R}} g = \lim_n \int_{\mathbb{R}} h_n = \lim_n \left(\int_{\{f=g\}} \varphi_n + \int_{\{f \neq g\}} \psi_n \right) = \lim_n \int_{\{f=g\}} \varphi_n.$$

Meanwhile,

$$\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} \varphi_n = \lim_n \left(\int_{\{f=g\}} \varphi_n + \int_{\{f \neq g\}} \varphi_n \right) = \lim_n \int_{\{f=g\}} \varphi_n,$$

and so $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$.

2. Take $\{\varphi_n\}, \{\psi_n\}$ as in the previous proof. Then $\{h_n := a\varphi_n + b\psi_n\}$ again a sequence of monotonically increasing simple functions with limit $af + bg$. Then

$$\int_{\mathbb{R}} (af + bg) = \lim_n \int_{\mathbb{R}} h_n = \lim_n \int_{\mathbb{R}} (a\varphi_n + b\psi_n) = \lim_n \left(a \int_{\mathbb{R}} \varphi_n + b \int_{\mathbb{R}} \psi_n \right) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g.$$

3. wlog, assume that $f \leq g$ everywhere by replacing f with $f \mathbb{1}_{\{f \leq g\}}$. Then, $\{\varphi : \text{simple}, \varphi \leq f\} \subseteq \{\varphi : \text{simple}, \varphi \leq g\}$ and so $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$.
4. i. \Leftarrow clear. Conversely, we would like to prove that if $A = \{f > 0\}, m(A) = 0$. Put $A_n := \{f \geq \frac{1}{n}\}$ for $n \geq 1$. Then, $A_n \uparrow$ and $\bigcup_{n=1}^{\infty} A_n = A$. By continuity of m ,

$$m(A) = \lim_n m(A_n).$$

Suppose towards a contradiction that $m(A) = \delta > 0$. Then, $\delta = \lim_n m(A_n)$, and so must exist $N \geq 1$ such that $m(A_N) \geq \frac{\delta}{2}$. Since $f \geq f \mathbb{1}_{A_N} \geq \frac{1}{N} \mathbb{1}_{A_N}$. By monotonicity, $\int_{\mathbb{R}} f \geq \int_{\mathbb{R}} \frac{1}{N} \mathbb{1}_{A_N} = \frac{1}{N} m(A_N) \geq \frac{1}{N} \frac{\delta}{2} > 0$, a contradiction.

ii. By i., $\int_A f = 0 \Leftrightarrow \mathbb{1}_A f \equiv 0$ a.e. on \mathbb{R} . If $m(A) = 0$, then $\mathbb{1}_A \equiv 0$ a.e. so $\mathbb{1}_A f \equiv 0$ a.e.. Else, if $m(A) > 0$, then $f \equiv 0$ a.e. on A .

iii. Put $A := \{f = \infty\}$. Assume towards a contradiction that $m(A) = \delta > 0$. Then, for every $n \geq 1, f \geq f \mathbb{1}_A \geq n \mathbb{1}_A$ and so $\int_{\mathbb{R}} f \geq \int_{\mathbb{R}} n \mathbb{1}_A = nm(A) = n\delta$. But this holds for any arbitrary n , so $\int_{\mathbb{R}} f = \infty$, a contradiction.

5. Put $A_a := \{f > a\}$. Then $f \geq f \mathbb{1}_{A_a} > a \mathbb{1}_{A_a}$ so $\int_{\mathbb{R}} f \geq am(A_a)$.

■

2.5.3 Integral of General Measurable, Integrable Functions

\Leftrightarrow **Definition 2.10:** For f measurable, $\int_{\mathbb{R}} f := \int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-$, provided that at least one of $\int_{\mathbb{R}} f^+, \int_{\mathbb{R}} f^-$ is finite; in particular, $\int_{\mathbb{R}} f$ may be finite or infinite.

Remark 2.7: Only having $\int_{\mathbb{R}} f$ being defined is not sufficient for the desirable properties (linearity, monotonicity) to hold.

↪ **Definition 2.11** (Integrable): A measurable function f is called *integrable*, denoted $f \in L^1(\mathbb{R})$, if both $\int_{\mathbb{R}} f^+ < \infty$ and $\int_{\mathbb{R}} f^- < \infty$. Note that

$$\begin{aligned} f \in L^1(\mathbb{R}) &\Leftrightarrow \int_{\mathbb{R}} |f| < \infty \text{ (since } \int_{\mathbb{R}} |f| = \int_{\mathbb{R}} f^+ + \int_{\mathbb{R}} f^-) \\ &\Leftrightarrow \int_{\mathbb{R}} f \text{ finite valued.} \end{aligned}$$

↪ **Proposition 2.21** (Properties of Integrals of Integrable Functions):

1. $|\int_{\mathbb{R}} f| \leq \int_{\mathbb{R}} |f|$
2. $f \in L^1(\mathbb{R}) \Rightarrow f$ is finite valued a.e.
3. (Linearity) For $f, g \in L^1(\mathbb{R})$ and $a, b \in \mathbb{R}$, $af + bg \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} (af + bg) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g$
4. If $f \in L^1(\mathbb{R})$ and $A \in \mathcal{M}$ and $m(A) = 0$ then $\int_A f = 0$; in particular if $f, g \in L^1(\mathbb{R})$ with $f = g$ a.e. then $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g$
5. (Monotonicity) If $f, g \in L^1(\mathbb{R})$ with $f \leq g$ a.e., then $\int_{\mathbb{R}} f \leq \int_{\mathbb{R}} g$

PROOF.

1. $-\int_{\mathbb{R}} f^- \leq \int_{\mathbb{R}} f \leq \int_{\mathbb{R}} f^+$ and $\int_{\mathbb{R}} f^{\pm} \leq \int_{\mathbb{R}} |f|$.
2. We know $\int_{\mathbb{R}} |f| < \infty$ so $|f| < \infty$ a.e. by properties of integrals of non-negative functions so $m(\{f = \pm\infty\}) = 0$
3. $|af| \leq |a| |f|$ so by monotonicity of non-negative functions, $\int_{\mathbb{R}} |af| \leq |a| \int_{\mathbb{R}} |f| < \infty$ so af in $L^1(\mathbb{R})$. Note then that

$$(af)^+ = \begin{cases} af^+ & \text{if } a \geq 0 \\ -af^- & \text{if } a < 0 \end{cases} \quad (af)^- = \begin{cases} af^- & \text{if } a \geq 0 \\ -af^+ & \text{if } a < 0 \end{cases}$$

so

$$\begin{aligned} \int_{\mathbb{R}} af &= \int_{\mathbb{R}} (af)^+ - \int_{\mathbb{R}} (af)^- \\ &= \begin{cases} \int_{\mathbb{R}} af^+ - \int_{\mathbb{R}} af^- & \text{if } a \geq 0 \\ \int_{\mathbb{R}} (-a)f^- - \int_{\mathbb{R}} (-a)f^+ & \text{if } a < 0 \end{cases} \\ &= \begin{cases} a(\int_{\mathbb{R}} f^+ - \int_{\mathbb{R}} f^-) & \text{if } a \geq 0 \\ (-a)(\int_{\mathbb{R}} f^- - \int_{\mathbb{R}} f^+) & \text{if } a < 0 \end{cases} = a \int_{\mathbb{R}} f. \end{aligned}$$

By the same argument $bg \in L^1(\mathbb{R})$ and $\int_{\mathbb{R}} (bg) = b \int_{\mathbb{R}} g$. wlog, $a = b = 1$. We want to show $f + g \in L^1(\mathbb{R})$; clearly $|f + g| \leq |f| + |g| < \infty$ so it must be $f + g \in L^1(\mathbb{R})$. Set $h := f + g$ then $|h, f, g| < \infty$ a.e. and each of the integrals of $|h, f, g| < \infty$. Then, $h^+ - h^- = f^+ - f^- + g^+ - g^-$. Then $h^+ + f^- + g^- = f^+ + g^+ + h^-$, where now both sides are non-negative functions. By linearity of integrals of non-negative functions and since all terms finite a.e.,

$$\begin{aligned} \int h^+ + \int f^- + \int g^- &= \int f^+ + \int g^+ + \int h^- \\ \Rightarrow \int h^+ - \int h^- &= \int f^+ - \int f^- + \int g^+ - \int g^- \\ \Rightarrow \int (f + g) &= \int h = \int f + \int g. \end{aligned}$$

4. $|\int_A f| \leq \int_A |f| = 0.$

5. Put $h = g - f$ (valid since $f, g \in L^1(\mathbb{R})$) then $h \geq 0$ a.e. Then $\int_{\mathbb{R}} h \geq 0$ so by linearity $\int_{\mathbb{R}} (g - f) = \int_{\mathbb{R}} g - \int_{\mathbb{R}} f \geq 0.$

■

§2.6 Convergence Theorems of Integral

↪ **Theorem 2.7** (Monotone Coverage Theorem (MON)): Assume $\{f_n\}, f$ are non-negative, measurable functions. If $f_n \uparrow$ and $\lim_{n \rightarrow \infty} f_n = f$, then

$$\int_{\mathbb{R}} f = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n.$$

Remark 2.8: When we write $\lim_n f_n = f$, we mean pointwise convergence; however, one can replace these statements with convergence a.e. and obtain an equivalent, more general result wlog.

PROOF. By monotonicity of non-negative functions, $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n$ exists, forming an increasing sequence. Since $f_n \leq f$, then we know too that $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f_n \leq \int_{\mathbb{R}} f.$

Conversely, for every n , let $\{\varphi_{n,k}\}_{k \in \mathbb{N}}$ be a sequence of simple functions such that $\varphi_{n,k} \uparrow$ w.r.t k and $\varphi_{n,k} \rightarrow f_n$ as $k \rightarrow \infty;$

$$\begin{array}{ccccccc} f_1 & f_2 & \cdots & f_k & f_{k+1} & \cdots & \rightarrow f \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \\ \varphi_{1,k} & \varphi_{2,k} & \ddots & \varphi_{k,k} & \varphi_{k+1,k} & \cdots & \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \\ \varphi_{1,2} & \varphi_{2,2} & \ddots & \varphi_{k,2} & \varphi_{k+1,2} & \cdots & \\ \varphi_{1,1} & \varphi_{2,1} & \cdots & \varphi_{k,1} & \varphi_{k+1,1} & \cdots & \end{array}$$

For each $k \geq 1$, let

$$g_k := \max\{\varphi_{1,k}, \varphi_{2,k}, \dots, \varphi_{k,k}\}.$$

Then, g_k simple for each k , and $g_k \uparrow$ and $g_k \leq f$. So, $\lim_{k \rightarrow \infty} g_k$ exists. Then, for all $n \geq 1$, $\lim_{k \rightarrow \infty} g_k \geq \lim_{k \rightarrow \infty} \varphi_{n,k} = f_n$ so $\lim_{k \rightarrow \infty} g_k \geq \lim_{n \rightarrow \infty} f_n = f$. Thus, $\lim_{k \rightarrow \infty} \int_{\mathbb{R}} g_k = \int_{\mathbb{R}} f$ by a previous theorem. Since $\forall k \geq 1, \varphi_{1,k}, \varphi_{2,k}, \dots, \varphi_{k,k} \leq f_k, g_k \leq f_k$ and thus by monotonicity $\int_{\mathbb{R}} g_k \leq \int_{\mathbb{R}} f_k \Rightarrow \int_{\mathbb{R}} f = \lim_{k \rightarrow \infty} \int_{\mathbb{R}} g_k \leq \lim_{k \rightarrow \infty} \int_{\mathbb{R}} f_k$ as desired. ■

↪ **Corollary 2.5:** If $\{f_n\}, f$ measurable functions such that $f_n \uparrow$ and $\lim_n f_n = f$ and $\int_{\mathbb{R}} f_1^- < \infty$, then $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$.

PROOF. Since $f_n \uparrow, f_n \geq f_1$ so $f \geq f_1$. Then, $f_n^- \leq f_1^-, f^- \leq f_1^-$, all of these are finite valued a.e., and $\int_{\mathbb{R}} f_n^- \leq \int_{\mathbb{R}} f_1^- < \infty$ and $\int_{\mathbb{R}} f^- \leq \int_{\mathbb{R}} f_1^- < \infty$. For each $n \geq 1$, set $\tilde{f}_n := f_n + f_1^- = f_n^+ - f_n^- + f_1^- \geq 0$, and $\tilde{f}_n \uparrow$ with $\lim_n \tilde{f}_n = f + f_1^- =: \tilde{f} \geq 0$. By MON, $\int_{\mathbb{R}} \tilde{f} = \lim_n \int_{\mathbb{R}} \tilde{f}_n$ so $\int_{\mathbb{R}} (f + f_1^-) = \lim_n \int_{\mathbb{R}} (f_n + f_1^-)$.

We have that $\tilde{f}_n = f_n + f_1^- = f_n^+ - f_n^- + f_1^- \Rightarrow \tilde{f}_n + f_n^- = f_n^+ + f_1^-$, which is valid since $f_n^- < \infty$ a.e.. By linearity, then,

$$\begin{aligned} \int_{\mathbb{R}} \tilde{f}_n + \int_{\mathbb{R}} f_n^- &= \int_{\mathbb{R}} f_n^+ + \int_{\mathbb{R}} f_1^- \\ \Rightarrow \int_{\mathbb{R}} \tilde{f}_n &= \int_{\mathbb{R}} f_n^+ - \int_{\mathbb{R}} f_n^- + \int_{\mathbb{R}} f_1^- \quad \text{because } \int_{\mathbb{R}} f_n^- < \infty \\ \Rightarrow \int_{\mathbb{R}} \tilde{f}_n &= \int_{\mathbb{R}} f_n + \int_{\mathbb{R}} f_1^-. \end{aligned}$$

Similar work gives $\int_{\mathbb{R}} \tilde{f} = \int_{\mathbb{R}} f + \int_{\mathbb{R}} f_1^-$, and taking limits and using $\lim_n \int_{\mathbb{R}} (f_n + f_1^-) = \int_{\mathbb{R}} (f + f_1^-)$ completes the proof. ■

↪ **Theorem 2.8 (Reverse MON):** Assume $\{f_n\}$, measurable such that $f_n \downarrow$ and $\lim_{n \rightarrow \infty} f_n = f$. If $\int_{\mathbb{R}} f_1^+ < \infty$, then $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$.

PROOF. Consider $\{-f_n\}$ and use the previous corollary. ■

↪ **Theorem 2.9 (Fatou's Lemma):** Assume $\{f_n\}$ non-negative, measurable. Then

$$\int_{\mathbb{R}} \left(\liminf_{n \rightarrow \infty} f_n \right) \leq \liminf_{n \rightarrow \infty} \left(\int_{\mathbb{R}} f_n \right).$$

PROOF. For every $m \geq 1$, set $g_m := \inf_{n \geq m} f_n$. Then, g_m non-negative and $g_m \uparrow$, with $\lim_m g_m = \liminf_n f_n$. By MON, $\int_{\mathbb{R}} \liminf_n f_n = \lim_{m \rightarrow \infty} \left(\int_{\mathbb{R}} g_m \right)$. For every $n \geq m$, $g_m \leq f_n$, so by monotonicity, $\int_{\mathbb{R}} g_m \leq \int_{\mathbb{R}} f_n$ for every $n \geq m$, so $\int_{\mathbb{R}} g_m \leq \inf_{n \geq m} \int_{\mathbb{R}} f_n$, and hence $\lim_{m \rightarrow \infty} \int_{\mathbb{R}} g_m \leq \lim_{m \rightarrow \infty} \inf_{n \geq m} \int_{\mathbb{R}} f_n = \liminf_n \left(\int_{\mathbb{R}} f_n \right)$, and the proof follows. ■

↪ **Corollary 2.6:** Assume $\{f_n\}$ measurable and there exists a measurable function g such that $\int_{\mathbb{R}} g^- < \infty$ and $f_n \geq g$ for every n . Then,

$$\int_{\mathbb{R}} \left(\liminf_n f_n \right) \leq \liminf_n \left(\int_{\mathbb{R}} f_n \right).$$

PROOF. Since $f_n \geq g$ for all $n \geq 1$, $f_n^- \leq g^-$ so $f_n^- < \infty$ a.e. and $\int_{\mathbb{R}} f_n^- < \infty$. Set $\tilde{f}_n := f_n + g^- \geq 0$. Then, apply Fatou to get $\int_{\mathbb{R}} \liminf_n \tilde{f}_n \leq \liminf_n \int_{\mathbb{R}} \tilde{f}_n$, then it suffices to check linearity. ■

↪ **Theorem 2.10** (Reverse Fatou): Assume $\{f_n\}$ measurable and there exists a g measurable such that $\int_{\mathbb{R}} g^+ < \infty$ and $f_n \leq g$ for all $n \geq 1$. Then,

$$\int_{\mathbb{R}} \left(\limsup_n f_n \right) \geq \limsup_n \left(\int_{\mathbb{R}} f_n \right).$$

PROOF. Apply previous proof to $\{-f_n\}$. ■

Remark 2.9: The “floor” g is necessary. Let $f_n(x) := \begin{cases} -1 & \text{if } x \geq n \\ 0 & \text{if } x < n \end{cases}$. Then, $f_n \uparrow$, and $\lim_n f_n = 0$ while $\int_{\mathbb{R}} f_n = -\infty$ for every n , so MON doesn’t apply.

↪ **Theorem 2.11** (Dominated Convergence Theorem (DOM)): Assume $\{f_n\}, f$ measurable with $\lim_n f_n = f$. If there exists a $g \in L^1(\mathbb{R})$ such that $|f_n| \leq |g|$ for all n , then $f_n \rightarrow f$ in $L^1(\mathbb{R})$ i.e. $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f_n - f| = 0$. In particular, $\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n$.

PROOF. Since $|f_n| \leq |g|$ and $f = \lim_{n \rightarrow \infty} f_n$, then $|f| \leq |g|$. So, $\int_{\mathbb{R}} |f_n| \leq \int_{\mathbb{R}} |g| < \infty$ and similarly $\int_{\mathbb{R}} |f| \leq \int_{\mathbb{R}} |g| < \infty$ so $|f_n|, f \in L^1(\mathbb{R})$.

Observe that $|f_n - f| \leq 2|g|$, and $\int_{\mathbb{R}} (2|g|) < \infty$. Applying Reverse Fatou to $\{|f_n - f|\}_{n \in \mathbb{N}}$, we find

$$\begin{aligned} \int_{\mathbb{R}} \left(\underbrace{\limsup_n (|f_n - f|)}_0 \right) &\geq \limsup_n \left(\int_{\mathbb{R}} |f_n - f| \right) \\ &\Rightarrow \lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f_n - f| = 0, \end{aligned}$$

so in particular

$$\left| \int_{\mathbb{R}} f_n - \int_{\mathbb{R}} f \right| = \left| \int_{\mathbb{R}} (f_n - f) \right| \leq \int_{\mathbb{R}} |f_n - f| \rightarrow 0$$

so $\lim_n \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$. ■

Remark 2.10: We must find $g \in L^1(\mathbb{R})$ to dominate $|g| \geq |f_n|$ irrespective of n . For instance, if $f_n = \mathbb{1}_{[n, 2n]}$, then $\lim_n f_n = 0$, but $\int_{\mathbb{R}} f_n = n$ for all $n \geq 1$. DOM doesn’t apply, since we would need a constant 1 function to dominate all f_n , which is not integrable.

↪ **Proposition 2.22:** Assume $f \in L^1(\mathbb{R}), \{h_n\}$ a sequence of measurable functions that are uniformly bounded, i.e. $\exists M > 0$ such that $|h_n| \leq M$ a.e. for all $n \geq 1$. If $h_n \rightarrow h$ a.e. for some measurable function h , then

$$\lim_n \int_{\mathbb{R}} (fh_n) = \int_{\mathbb{R}} (fh).$$

PROOF. For every n , $|f \cdot h_n| \leq M |f| \in L^1(\mathbb{R})$. The conclusion follows from DOM. ■

↪ **Corollary 2.7:** If $f \in L^1(\mathbb{R})$ then for all $\varepsilon > 0$, there exists a compact set $K \subseteq \mathbb{R}$ such that $\int_{K^c} |f| \leq \varepsilon$.

PROOF. If $h_n := \mathbb{1}_{[-n,n]}$, the $\lim_n \int_{\mathbb{R}} f h_n = \lim_n \int_{[-n,n]} f = \int_{\mathbb{R}} f$, and also $\lim_n \int_{\{\mathbb{R}-[-n,n]\}} f = 0$. ■

↪ **Corollary 2.8:** If $f \in L^1(\mathbb{R})$, then for all $\varepsilon > 0$, $\exists N \geq 1$ such that $\int_{\{|f| > N\}} |f| \leq \varepsilon$.

PROOF. Let $h_n = \mathbb{1}_{\{|f| > n\}}$ then $\lim_{n \rightarrow \infty} \int_{\{|f| > n\}} f = 0$. ■

↪ **Corollary 2.9:** If $\{A_n\} \subseteq \mathcal{M}$ such that $A_n \uparrow$, then $\int_{\cup_{n=1}^{\infty} A_n} f = \lim_{n \rightarrow \infty} \int_{A_n} f$ ($\mathbb{1}_{A_n} f \rightarrow \mathbb{1}_{\cup_{n=1}^{\infty} A_n} f$).

↪ **Corollary 2.10** (Countable Additivity): If $\{B_n\} \subseteq \mathcal{M}$ are disjoint, then $\int_{\cup_{n=1}^{\infty} B_n} f = \sum_{n=1}^{\infty} \int_{B_n} f$.

↪ **Corollary 2.11:** If $\{A_n\} \subseteq \mathcal{M}$ such that $A_n \downarrow$, then $\int_{\cap_{n=1}^{\infty} A_n} f = \lim_{n \rightarrow \infty} \int_{A_n} f$.

↪ **Proposition 2.23:** Assume f is non-negative, measurable, and finite-valued a.e.. Then, for every $k \in \mathbb{Z}$, put $A_k := \{x \in \mathbb{R} : 2^k \leq f(x) < 2^{k+1}\}$. Then,

$$f \text{ integrable} \Leftrightarrow \int_{\mathbb{R}} f < \infty \Leftrightarrow \sum_{k \in \mathbb{Z}} 2^k m(A_k) < \infty.$$

PROOF. (\Rightarrow) Note that the A_k 's disjoint and $\cup_{k \in \mathbb{Z}} A_k = \{0 < f < \infty\}$. So,

$$\int_{\mathbb{R}} f = \underbrace{\int_{\{f=0\}} f}_{=0 \text{ since } f=0} + \int_{\{0 < f < \infty\}} + \underbrace{\int_{\{f=\infty\}} f}_{=0 \text{ since } f < \infty \text{ a.e.}} = \sum_{k \in \mathbb{Z}} \int_{A_k} f.$$

For each $k \in \mathbb{Z}$, for every $x \in A_k$, $2^k \leq f(x) < 2^{k+1}$ so $2^k m(A_k) \leq \int_{A_k} f(x) < 2^{k+1} m(A_k)$. Hence,

$$\sum_{k \in \mathbb{Z}} 2^k m(A_k) \leq \sum_{k \in \mathbb{Z}} \int_{A_k} f = \int_{\mathbb{R}} f < \infty.$$

(\Leftarrow) Suppose $\sum_{k \in \mathbb{Z}} 2^k m(A_k) < \infty$. We know again

$$\int_{\mathbb{R}} f = \int_{\{0 < f < \infty\}} f \stackrel{\text{By } \overline{\text{MON}}}{=} \sum_{k \in \mathbb{Z}} \int_{A_k} f < \sum_{k \in \mathbb{Z}} 2^{k+1} m(A_k) = 2 \sum_{k \in \mathbb{Z}} 2^k m(A_k) < \infty.$$

⊗ **Example 2.3:** Let $f(x) = |x|^{-\alpha} \mathbb{1}_{[-1,1]}(x)$, with $f(0) = \infty$ and $\alpha > 0$; f finite-valued a.e.. For every $k \in \mathbb{Z}$, put $A_k := \{2^k \leq f < 2^{k+1}\} = \{x \in [-1, 1] : 2^k \leq |x|^{-\alpha} < 2^{k+1}\}$. By definition, $|f| \geq 1$, so

$$A_k = \left[-2^{-\frac{k}{\alpha}}, -2^{-\frac{(k+1)}{\alpha}}\right) \cup \left(2^{-\frac{(k+1)}{\alpha}}, 2^{-\frac{k}{\alpha}}\right] \text{ for } k \geq 0, \quad A_k = \emptyset \text{ if } k < 0.$$

Hence,

$$\sum_{k \in \mathbb{Z}} 2^k m(A_k) = \sum_{k=0}^{\infty} 2^k \cdot 2 \cdot \left(1 - 2^{-\frac{1}{\alpha}}\right) 2^{-\frac{k}{\alpha}} = 2 \left(1 - 2^{-\frac{1}{\alpha}}\right) \sum_{k=0}^{\infty} 2^{k(1-\frac{1}{\alpha})}.$$

Hence, the series $< \infty \Leftrightarrow \alpha < 1$, and thus $\int_{[-1,1]} |x|^{-\alpha} dx < \infty \Leftrightarrow \alpha < 1$.

⊗ **Example 2.4:** Let $g(x) = |x|^{-\beta} \mathbb{1}_{\mathbb{R}-[-1,1]}(x)$ with $\beta > 0$. We have $|g| < 1$; we again put

$$A_k := \{2^k \leq g < 2^{k+1}\} = \begin{cases} \left[-2^{-\frac{k}{\beta}}, -2^{-\frac{(k+1)}{\beta}}\right) \cup \left(2^{-\frac{(k+1)}{\beta}}, 2^{-\frac{k}{\beta}}\right] & \text{if } k < 0 \\ \emptyset & \text{if } k \geq 0. \end{cases}$$

So,

$$\int_{\mathbb{R}-[-1,1]} |x|^{-\beta} dx < \infty \Leftrightarrow \sum_{k \in \mathbb{Z}} 2^k m(A_k) < \infty \Leftrightarrow \beta > 1.$$

⊗ **Example 2.5:** Let $f_n(x) = \left(1 + \frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)$. What is $\lim_{n \rightarrow \infty} \int_{(0,\infty)} f_n(x) dx$? We have that for all $x > 0$, $\lim_{n \rightarrow \infty} f_n(x) = 0$. We have that since $|\sin\left(\frac{x}{n}\right)| \leq 1$, so

$$|f_n(x)| \leq \left(1 + \frac{x}{n}\right)^{-n} \leq \left(1 + \frac{x}{2}\right)^{-2} \quad \forall x > 0, \forall n \geq 2.$$

Let $g(x) := \left(1 + \frac{x}{2}\right)^{-2}$. We would like to apply DOM, so we need to check that $g \in L^1((0, \infty))$. We have that

$$\int_{(0,\infty)} g = \int_{(0,1]} g + \int_{(1,\infty)} g \leq \int_{(0,1]} 1 + \underbrace{\int_{(1,\infty)} \frac{4}{x^2} dx}_{\beta=2 \text{ of previous example}} < \infty,$$

so indeed $g \in L^1((0, \infty))$. Applying DOM, then, we have that

$$\lim_{n \rightarrow \infty} \int_{(0,\infty)} f_n = \int_{(0,\infty)} \lim_{n \rightarrow \infty} f_n = 0.$$

⊗ **Example 2.6:** Let $c > 0, f_n(x) = x^{-c}(\cosh x)^{-\frac{1}{n}}$. What is $\lim_n \int_{(1,\infty)} f_n$?

For every $x > 1, \cosh x > 1$, so $(\cosh x)^{-\frac{1}{n}} \uparrow$ with respect to n , with $\lim_n (\cosh x)^{-\frac{1}{n}} = 1$, so $\lim_{n \rightarrow \infty} f_n(x) = x^{-c}$ for every $x > 1$. Let $g(x) = x^{-c}$, then. By previous examples, when $c > 1, g \in L^1((1, \infty))$ so DOM applies and thus

$$\lim_n \int_{(1,\infty)} f_n = \int_{(1,\infty)} \lim_n f_n = \int_{(1,\infty)} x^{-c} dx < \infty.$$

When $0 < c \leq 1$, by Fatou,

$$\lim_n \inf \int_{(1,\infty)} f_n \geq \int_{(1,\infty)} \lim_n \inf(f_n) = \int_{(1,\infty)} x^{-c} dx,$$

since f_n converges. When $0 < c \leq 1$, the RHS = ∞ , and thus $\lim_{n \rightarrow \infty} \int_{(1,\infty)} f_n = \infty$.

⊗ **Example 2.7:** Let $c \geq 0, f_n(x) := \frac{n}{1+n^2x^2}$ for $x \geq 0$. What is $\lim_n \int_{[c,\infty)} f_n$?

We have that

$$\lim_n f_n(x) = \begin{cases} 0 & \text{if } x > 0 \\ \infty & \text{if } x = 0 \end{cases}$$

On $x \in [1, \infty), f_n(x) \geq f_{n+1}(x)$ for all $n \geq 1$, namely $f_n \downarrow$, and so $f_n(x) \leq f_1(x) = \frac{1}{1+x^2} \cdot f_1(x) \in L^1(\mathbb{R})$, by comparison with $\frac{1}{x^2}$ ($\alpha = 2$).

If $x \in (0, 1), f_n(x) = \frac{1}{x} \frac{nx}{1+(nx)^2} \leq A \frac{1}{x}$, with $A := \sup_{t>0} \frac{t}{1+t^2} < \infty$. But $\frac{A}{x} \notin L^1((0, 1))$.

When $c > 0$, for all $x \geq c$ and for all $n \geq 1$,

$$f_n(x) \leq \mathbb{1}_{[1,\infty)}(x) \frac{1}{1+x^2} + \mathbb{1}_{[c,1)}(x) \frac{A}{x} \leq \mathbb{1}_{[1,\infty)}(x) \frac{1}{1+x^2} + \mathbb{1}_{[c,1)}(x) \frac{A}{c} \in L^1([c, \infty)).$$

Hence, we may apply DOM, so

$$\lim_n \int_{[c,\infty)} f_n = \int_{[c,\infty)} \lim_n f_n = 0,$$

when $c > 0$. However, when $c = 0$, we have no such dominating function; so what is

$$\int_{[0,\infty)} f_n(x) dx?$$

§2.7 Riemann Integral vs Lebesgue Integral

Recall; let f be bounded on $[a, b]$. Then, f is Riemann integrable on $[a, b]$ if

$$\begin{cases} f \text{ is continuous on } [a, b] \\ f \text{ is monotonic on } [a, b] \\ f \text{ is continuous except at possibly finitely many points in } [a, b] \end{cases}.$$

Recall the function $f = \mathbb{1}_{\mathbb{Q} \cap [0,1]}$. f is not Riemann integrable, but is Lebesgue integrable, because $|f| \leq \mathbb{1}_{[0,1]} \in L^1(\mathbb{R})$.

Remark 2.11:

1. \exists bounded functions on $[a, b]$ that are not Riemann integrable.
2. In general, g being Riemann integrable and $|f| \leq |g| \not\Rightarrow f$ is Riemann integrable ($\mathbb{1}_{\mathbb{Q} \cap [0,1]} \leq \mathbb{1}_{[0,1]}$).
3. In general, DOM and MON do *not* apply to Riemann integrable. For instance, consider $\{q_n\}$ an enumeration of $\mathbb{Q} \cap [0, 1]$. Define $f_n(x) := \begin{cases} 1 & \text{if } x \in \{q_1, \dots, q_n\} \\ 0 & \text{else} \end{cases}$. $f_n \uparrow$, with $f_n \rightarrow \mathbb{1}_{\mathbb{Q} \cap [0,1]}$. So, MON applies with the Lebesgue integral, but f_n is only discontinuous, for every n , at finitely many points, so f_n Riemann integrable with $\int_0^1 f_n = 0$, but the limit is not Riemann integrable.

↔ Theorem 2.12: Assume f is Riemann integrable on $[a, b]$. Then, f is Lebesgue integrable on $[a, b]$, i.e. $f \in L^1([a, b])$. Moreover, $\int_a^{b^{(R)}} f = \int_{[a,b]} f$.

PROOF. f is Riemann integrable on $[a, b]$, so there is some $M > 0$ such that $|f| \leq M$ on $[a, b]$. Further, there exist step functions φ_n, ψ_n with $\varphi_n \leq f \leq \psi_n$ on $[a, b]$ and $|\varphi_n|, |\psi_n| \leq M$ for all $n \geq 1$, and

$$\lim_{n \rightarrow \infty} \int_a^{b^{(R)}} \varphi_n = \int_a^{b^{(R)}} f = \lim_{n \rightarrow \infty} \int_a^{b^{(R)}} \psi_n.$$

Denote $\varphi := \lim_{n \rightarrow \infty} \varphi_n, \psi := \lim_{n \rightarrow \infty} \psi_n$, which exist by Monotonicity. Since φ_n, ψ_n are step functions, they are measurable hence φ, ψ measurable with $\varphi \leq f \leq \psi$. Observe that the Lebesgue, Riemann integral coincide on step functions. Hence, $\int_a^{b^{(R)}} \varphi_n = \int_{[a,b]} \varphi_n$, same with ψ_n . By DOM, (with M as the dominator)

$$\int_{[a,b]} \varphi = \lim_n \int_{[a,b]} \varphi_n = \lim_n \int_a^{b^{(R)}} \varphi_n = \int_a^{b^{(R)}} (f) = \lim_n \int_a^{b^{(R)}} \psi_n = \lim_n \int_{[a,b]} \psi_n = \int_{[a,b]} \psi.$$

Since $\varphi \leq \psi$ and $\int_{[a,b]} (\psi - \varphi) = 0$, we have that $\psi = \varphi$ a.e. on $[a, b]$ by properties of integrals of non-negative functions, and thus $f = \varphi = \psi$ a.e. on $[a, b]$. In particular, then, f is measurable, being equal a.e. to measurable functions. Thus, since $|f| \leq M$ on $[a, b], f \in L^1([a, b])$, and so since integrals agree on functions that are equal a.e., $\int_{[a,b]} f = \int_{[a,b]} \varphi = \int_a^{b^{(R)}} f$ as desired. ■

⊗ **Example 2.8:** We return to our example of computing $\lim_{n \rightarrow \infty} \int_{[0, \infty)} \frac{n}{1+n^2x^2} dx$. We may rewrite

$$\int_{[0, \infty)} \frac{n}{1+n^2x^2} dx = \int_{[0, T]} \frac{n}{1+n^2x^2} dx + \int_{[T, \infty)} \frac{n}{1+n^2x^2} dx$$

where $T > 0$. We know from the previous example that the RHS integral converges to 0 by application of DOM. Now, $\frac{n}{1+n^2x^2}$ is continuous on $[0, T]$ and thus Riemann integrable, and so by the previous theorem

$$\int_{[0, T]} \frac{n}{1+n^2x^2} = \int_{[0, T]}^{(R)} \frac{n}{1+n^2x^2} = \arctan(nT).$$

As $n \rightarrow \infty$, $\arctan(nT) \rightarrow \frac{\pi}{2}$, and thus the limit of the whole integral indeed exists, and is in fact equal to $\frac{\pi}{2}$.

§2.8 L^p -space

↔ **Definition 2.12** (p -integrable): Let f measurable and $1 \leq p < \infty$. We say f is p -integrable and write $f \in L^p(\mathbb{R})$ if $\int_{\mathbb{R}} |f|^p < \infty$, i.e. $|f|^p \in L^1(\mathbb{R})$.

For $f \in L^p(\mathbb{R})$, define the p -norm

$$\|f\|_p := \left(\int_{\mathbb{R}} |f|^p \right)^{\frac{1}{p}}.$$

Remark 2.12: When $p = 1$, we see that $\|\cdot\|_1$ a norm fairly clearly from properties of the integral. We need to show this for more general $p > 1$.

Remark 2.13: $\|\cdot\|_p$ also defined when $p = \infty$; given f measurable, we define

$$\|f\|_{\infty} := \text{ess sup}_{x \in \mathbb{R}} |f(x)| := \inf \{ a \in \overline{\mathbb{R}} : |f| \leq a \text{ a.e.} \}.$$

Then, we define

$$L^{\infty}(\mathbb{R}) := \{f \text{ measurable s.t. } \|f\|_{\infty} < \infty\}.$$

One can show that if $f \in L^{\infty}(\mathbb{R})$, $|f| \leq \|f\|_{\infty}$ a.e..

↔ **Theorem 2.13** (Hölder's Inequality): Let $1 < p < \infty$ and let $q := \frac{p}{p-1}$ (such a q is called the Hölder Conjugate of p). If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then $fg \in L^1(\mathbb{R})$, and

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

In particular, if $p = q = 2$, then we have the *Cauchy-Schwarz Inequality*.

Remark 2.14: $\frac{1}{p} + \frac{1}{q} = 1$.

PROOF. We will employ “Young’s Inequality”, which states that for all $a, b \geq 0$, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ where $\frac{1}{p} + \frac{1}{q} = 1$. Since $f \in L^p, g \in L^q$, set $\tilde{f} := \frac{f}{\|f\|_p}$ and $\tilde{g} := \frac{g}{\|g\|_q}$. Then, a.e.

$$|\tilde{f}\tilde{g}| \leq \frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q}.$$

We have

$$\int_{\mathbb{R}} |\tilde{f}\tilde{g}| = \int_{\mathbb{R}} \frac{|fg|}{\|f\|_p \|g\|_q}$$

and

$$\int_{\mathbb{R}} \frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q} = \frac{1}{p} \frac{\int_{\mathbb{R}} |f|^p}{\|f\|_p^p} + \frac{1}{q} \frac{\int_{\mathbb{R}} |g|^q}{\|g\|_q^q} = \frac{1}{p} + \frac{1}{q} = 1,$$

and thus

$$\int_{\mathbb{R}} |fg| = \|fg\|_q \leq \|f\|_p \|g\|_q$$

as required. ■

Remark 2.15: This inequality also holds for $p = 1, q = \infty$ (assignment question).

↪ **Lemma 2.2:** For all $a, b \geq 0$, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ where $\frac{1}{p} + \frac{1}{q} = 1$.

PROOF. ■

↪ **Theorem 2.14** (Minkowski's Inequality): Let $1 \leq p < \infty$ and $f, g \in L^p(\mathbb{R})$. Then, $f + g \in L^p(\mathbb{R})$, and in particular

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p.$$

In particular, then, $\|\cdot\|_p$ satisfies the triangle inequality and is indeed a norm on $L^p(\mathbb{R})$.

PROOF. We have $|f + g|^p \leq 2^p(|f|^p + |g|^p)$ hence $f + g \in L^p(\mathbb{R})$ since $|f|^p, |g|^p \in L^1(\mathbb{R})$. Further

$$\begin{aligned}
\int_{\mathbb{R}} |f + g|^p &= \int_{\mathbb{R}} |f + g| |f + g|^{p-1} \leq \int_{\mathbb{R}} |f| |f + g|^{p-1} + \int_{\mathbb{R}} |g| |f + g|^{p-1} \\
\text{(Hölder's)} \quad &\leq \left(\int_{\mathbb{R}} |f|^p \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |f + g|^{(p-1)q} \right)^{\frac{1}{q}} + \left(\int_{\mathbb{R}} |g|^p \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |f + g|^{(p-1)q} \right)^{\frac{1}{q}} \\
&\leq (\|f\|_p + \|g\|_p) \left(\int_{\mathbb{R}} |f + g|^p \right)^{\frac{1}{q}} \\
\Rightarrow \|f + g\|_p &= \left(\int_{\mathbb{R}} |f + g|^p \right)^{\frac{1}{p}} = \left(\int_{\mathbb{R}} |f + g|^p \right) \cdot \left(\int_{\mathbb{R}} |f + g|^p \right)^{-\frac{1}{q}} \\
&\leq (\|f\|_p + \|g\|_p) \left(\int_{\mathbb{R}} |f + g|^p \right)^{\frac{1}{q}} \cdot \left(\int_{\mathbb{R}} |f + g|^p \right)^{-\frac{1}{q}} = \|f\|_p + \|g\|_p \\
\Rightarrow \|f + g\|_p &\leq \|f\|_p + \|g\|_p
\end{aligned}$$

■

Remark 2.16: Minkowski's also holds for $p = \infty$.

↪ **Lemma 2.3:** Let $1 \leq p < \infty$. If $\{g_k\} \in L^p(\mathbb{R})$ such that $\sum_{k=1}^{\infty} \|g_k\|_p < \infty$, then $\exists G \in L^p(\mathbb{R})$ such that $G_m := \sum_{k=1}^m g_k \rightarrow G$ as $m \rightarrow \infty$ a.e. as well as in $L^p(\mathbb{R})$.

PROOF. Put $\widetilde{G}_m := \sum_{k=1}^m |g_k|$ and $\widetilde{G} := \sum_{k=1}^{\infty} |g_k|$. Then, $\widetilde{G}_m \uparrow$ with $\lim_{m \rightarrow \infty} \widetilde{G}_m = \widetilde{G}$. By MON,

$$\int_{\mathbb{R}} \widetilde{G}^p = \lim_{m \rightarrow \infty} \int_{\mathbb{R}} \widetilde{G}_m^p = \lim_{m \rightarrow \infty} \|\widetilde{G}_m\|_p^p \leq \lim_{m \rightarrow \infty} \left(\sum_{k=1}^m \|g_k\|_p \right)^p$$

where the final inequality is by Minkowski's. Then,

$$\leq \left(\lim_{m \rightarrow \infty} \sum_{k=1}^m \|g_k\|_p \right)^p = \left(\sum_{k=1}^{\infty} \|g_k\|_p \right)^p < \infty, \text{ by assumption}$$

Hence, $\widetilde{G} \in L^p(\mathbb{R})$ and $\|\widetilde{G}\|_p \leq \sum_{k=1}^{\infty} \|g_k\|_p$ and thus \widetilde{G} finite-valued a.e. and hence $\sum_{k=1}^{\infty} g_k$ absolutely convergent a.e.. Set $G = \lim_{m \rightarrow \infty} G_m = \sum_{k=1}^{\infty} g_k$ a.e.. Moreover, we know

$$|G| = \left| \sum_{k=1}^{\infty} g_k \right| \leq \sum_{k=1}^{\infty} |g_k| = \widetilde{G} \Rightarrow G \in L^p(\mathbb{R})$$

and

$$|G - G_m| \leq \sum_{k=m+1}^{\infty} |g_k|.$$

Fix $\varepsilon > 0$. Since $\sum_{k=1}^{\infty} \|g_k\|_p < \infty$, exists some $M \geq 1$ such that $\sum_{k=M+1}^{\infty} \|g_k\|_p < \varepsilon$. Then,

$$\begin{aligned}
\int_{\mathbb{R}} |G - G_M|^p &\leq \int_{\mathbb{R}} \left(\sum_{k=M+1}^{\infty} |g_k| \right)^p = \lim_{L \rightarrow \infty} \int_{\mathbb{R}} \left(\sum_{k=M+1}^L |g_k| \right)^p \\
&\stackrel{\text{(Minkowski)}}{\leq} \lim_{L \rightarrow \infty} \left(\sum_{k=M+1}^L \|g_k\|_p \right)^p \\
&= \left(\sum_{k=M+1}^{\infty} \|g_k\|_p \right)^p \leq \varepsilon
\end{aligned}$$

hence $G_m \rightarrow G$ in $L^p(\mathbb{R})$. ■

↪ **Theorem 2.15:** Let $1 \leq p < \infty$. Then $L^p(\mathbb{R})$ is a complete normed space under the p -norm.

PROOF. Let $f_n \in L^p(\mathbb{R})$ be a Cauchy sequence under $\|\cdot\|_p$. We can choose a subsequence $\{n_k\}$ such that for every $k \geq 1$, $\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k}$. Set $g_k := f_{n_{k+1}} - f_{n_k}$. By the lemma, if $G_m := \sum_{k=1}^m g_k$, there exists some $G \in L^p(\mathbb{R})$ such that $G_m \rightarrow G$ a.e. and in $L^p(\mathbb{R})$. In fact, we have

$$G_m = \sum_{k=1}^m g_k = \sum_{k=1}^m (f_{n_{k+1}} - f_{n_k}) = f_{n_{m+1}} - f_{n_1},$$

hence

$$G = \lim_{m \rightarrow \infty} G_m = \left(\lim_{m \rightarrow \infty} f_{n_{m+1}} \right) - f_{n_1}.$$

Let $f := G + f_{n_1}$. Then, $f = \lim_{m \rightarrow \infty} f_{n_m}$ a.e. and since $G_m \rightarrow G$ in L^p , we have that $f_{n_m} \rightarrow f$ in L^p as $m \rightarrow \infty$. It remains to show convergence in L^p along the whole subsequence.

Fix $\varepsilon > 0$. Let $N \geq 1$ such that $\sup_{k, \ell \geq N} \|f_k - f_\ell\|_p < \varepsilon$ and m sufficiently large such that $n_m > N$ and $\|f_{n_m} - f\|_p \leq \varepsilon$. Then,

$$\|f_n - f\|_p \leq \underbrace{\|f_n - f_{n_m}\|_p}_{< \varepsilon} + \underbrace{\|f_{n_m} - f\|_p}_{< \varepsilon} < 2\varepsilon,$$

completing the proof. ■

Remark 2.17: L^∞ also complete.

2.8.1 Dense Subspaces of $L^p(\mathbb{R})$

↪ **Lemma 2.4:** Bounded and compactly supported functions are dense in $L^p(\mathbb{R})$.

PROOF. Given $f \in L^p(\mathbb{R})$, set

$$f_n(x) = \mathbb{1}_{[-n, n]}(x) \cdot f(x) \cdot \mathbb{1}_{\{|f| \leq n\}}(x)$$

which are bounded and compactly supported on $[-n, n]$. We claim $f_n \rightarrow f$ in $L^p(\mathbb{R})$.

We have $\int_{\mathbb{R}} |f_n - f|^p$ nonzero only if $x \notin [-n, n]$ or $|f(x)| > n$. Hence

$$\int_{\mathbb{R}} |f_n - f|^p \leq \int_{\mathbb{R} \setminus [-n, n]} |f|^p + \int_{\{|f| > n\}} |f|^p \rightarrow 0 \text{ as } n \rightarrow \infty.$$

■

↪ **Lemma 2.5:** Simple functions are dense in $L^p(\mathbb{R})$.

PROOF. For $f \in L^p(\mathbb{R})$, let f_n be as in the previous proof. For each $n \geq 1, k = 0, 1, \dots, n2^n - 1$, set

$$A_{n,k} := \left\{ x \in [-n, n] : \frac{k}{2^n} \leq f_n^+ < \frac{k+1}{2^n} \right\}, \quad \varphi_n^+ := \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{A_{n,k}},$$

and

$$B_{n,k} := \left\{ x \in [-n, n] : \frac{k}{2^n} \leq f_n^- < \frac{k+1}{2^n} \right\}, \quad \varphi_n^- := \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{B_{n,k}}.$$

Put $\varphi_n := \varphi_n^+ - \varphi_n^-$. This is a simple function, and $|\varphi_n| \leq n$ and supported on $[-n, n]$ for every n hence $\varphi_n \in L^p(\mathbb{R})$. In addition, $\lim_n \varphi_n(x) = f(x)$. In particular, for any $n \geq 1$,

$$|f_n(x) - \varphi_n(x)| \leq |f_n^+(x) - \varphi_n^+(x)| + |f_n^-(x) - \varphi_n^-(x)| \leq 2 \cdot 2^{-n}.$$

Then, in particular

$$\begin{aligned} \|f - \varphi_n\|_p &\leq \underbrace{\|f - f_n\|_p}_{\rightarrow 0} + \underbrace{\|f_n - \varphi_n\|_p}_{= \left(\int_{[-n, n]} |f_n - \varphi_n|^p \right)^{\frac{1}{p}}} \\ &\leq \left((2 \cdot 2^{-n})^p m([-n, n]) \right)^{\frac{1}{p}} \rightarrow 0 \end{aligned}$$

and so indeed $\varphi_n \rightarrow f$ in $L^p(\mathbb{R})$.

■

↪ **Theorem 2.16:** Let $C_c(\mathbb{R})$ denote the space of continuous and compactly supported functions. Then, $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

PROOF. Give $f \in L^p(\mathbb{R})$, let $\{\varphi_n\}$ simple functions as in the previous proof. Recall that, for every $n \geq 1$, there exists a step function θ_n such that $\theta_n \leq \sup_x |\varphi_n(x)| \leq n$, is supported on $[-n-1, n+1]$, and $\{\theta_n \neq \varphi_n\}$ has arbitrarily small measure. In particular, we choose θ_n such that $m(\{\theta_n \neq \varphi_n\}) \leq 2^{-n-1}$ for every $n \geq 1$.

Recall that given a step function θ_n , there exists a function $\tilde{\theta}_n$ continuous on \mathbb{R} , $\tilde{\theta}_n$ is supported on $[-n-2, n+2]$, and $m(\{\tilde{\theta}_n - \theta_n\}) \leq 2^{-n-1}$. Thus, $\{\tilde{\theta}_n\} \subseteq C_c(\mathbb{R})$, and

$$m(\{\tilde{\theta}_n - \varphi_n\}) \leq m(\{\tilde{\theta}_n - \theta_n\}) + m(\{\theta_n \neq \varphi_n\}) \leq 2^{-n}.$$

So, we have

$$\begin{aligned} \|f - \tilde{\theta}_n\|_p &\leq \underbrace{\|f - \varphi_n\|_p}_{\rightarrow 0 \text{ by lemma}} + \underbrace{\|\varphi_n - \tilde{\theta}_n\|_p} \\ &= \left(\int_{\mathbb{R}} |\varphi_n - \tilde{\theta}_n|^p \right)^{\frac{1}{p}} \\ &= \left(\int_{\{\tilde{\theta}_n \neq \varphi_n\}} |\varphi_n - \tilde{\theta}_n|^p \right)^{\frac{1}{p}} \\ &\leq (2^n)^{\frac{1}{p}} 2^{-n} \rightarrow 0 \end{aligned}$$

and thus $\tilde{\theta}_n \rightarrow f$ in $L^p(\mathbb{R})$. ■

Remark 2.18: The density of $C_c(\mathbb{R})$ in $L^p(\mathbb{R})$ is useful in the study of properties of generic L^p functions. For instance, show that if $f \in L^p(\mathbb{R})$, then $\lim_{n \rightarrow \infty} \int_{\mathbb{R}} |f(x + \frac{1}{n}) - f(x)|^p dx = 0$, that is $f(\cdot + \frac{1}{n}) \rightarrow f$ in $L^p(\mathbb{R})$ using this density.

Remark 2.19: $C_c(\mathbb{R})$ is NOT dense in $L^\infty(\mathbb{R})$.

§2.9 Convergence Modes and Uniform Integrability

Recall that, given $\{f_n\}, f$ measurable and finite-valued a.e., we have the following notions of convergence

1. $f_n \rightarrow f$ in measure $\Rightarrow \exists \{n_k\}$ such that $f_{n_k} \rightarrow f$ a.e. as $k \rightarrow \infty$
2. $f_n \rightarrow f$ a.e. on $A \in \mathcal{M}$ with $m(A) < \infty \Rightarrow f_n \rightarrow f$ in measure on A
3. $f_n \rightarrow f$ in $L^p(\mathbb{R})$.

Proposition 2.24: If $\{f_n\}, f$ in $L^p(\mathbb{R})$ for $1 \leq p < \infty$ and $f_n \rightarrow f$ in $L^p(\mathbb{R})$, then $f_n \rightarrow f$ in measure.

PROOF. For $\delta > 0$, we have

$$m(\{|f_n - f| > \delta\}) = \int_{\{|f_n - f| > \delta\}} 1 dx.$$

Remark that $1 \leq \frac{|f_n - f|}{\delta}$ over $\{|f_n - f| > \delta\}$; further $1^p = 1 \leq \left(\frac{|f_n - f|}{\delta}\right)^p$. Hence,

$$\leq \int_{\{|f_n - f| > \delta\}} \frac{|f_n - f|^p}{\delta^p} dx \leq \frac{1}{\delta^p} \int_{\mathbb{R}} |f_n - f|^p \leq \frac{1}{\delta^p} \|f_n - f\|_p^p.$$

But by assumption $\|f_n - f\|_p^p \rightarrow 0$ for any $\delta > 0$, hence $m(\{|f_n - f| > \delta\}) \rightarrow 0$ i.e. $f_n \rightarrow f$ in measure. ■

Remark 2.20: In general, convergence in $L^p \not\Rightarrow$ convergence a.e., with the same counter example from convergence in measure $\not\Rightarrow$ convergence a.e..

Remark 2.21: When do we have convergence a.e. \Rightarrow convergence in L^p ? This doesn't hold in general, unless some integral convergence theorem from before holds.

Remark 2.22: When do we have convergence in measure \Rightarrow convergence in L^p ? No in general, unless one of the integral convergence theorem holds; with some slight adaptation.

\hookrightarrow **Proposition 2.25** (MON, Measure Version (mMON)): Let f_n non-negative with $f_n \uparrow$ and $f_n \rightarrow f$ in measure. Then,

$$\int_{\mathbb{R}} f = \lim_n \int_{\mathbb{R}} f_n.$$

PROOF. $f_n \rightarrow f$ in measure implies $f_{n_k} \rightarrow f$ almost everywhere along some subsequence n_k , so it must be that f non-negative. Suppose the claim fails. Then, there exists some subsequence $\{n_\ell\}$ such that $\int_{\mathbb{R}} f_{n_\ell} \not\rightarrow \int_{\mathbb{R}} f$. However, along this subsequence we also have $f_{n_\ell} \rightarrow f$ in measure, and hence exists a subsubsequence n_{ℓ_p} such that $f_{n_{\ell_p}} \rightarrow f$ a.e.. Then, by MON applied to this subsubsequence, we know that

$$\lim_p \int_{\mathbb{R}} f_{n_{\ell_p}} = \int_{\mathbb{R}} f,$$

a contradiction. ■

\hookrightarrow **Proposition 2.26** (mDOM): If $f_n \in L^1(\mathbb{R})$ with $f_n \rightarrow f$ in measure and there exists some $g \in L^1(\mathbb{R})$ such that $|f_n| \leq |g|$, then $f_n \rightarrow f$ in $L^1(\mathbb{R})$.

Recall that if $f \in L^1(\mathbb{R})$, then $\int_{\{|f|>n\}} |f| \rightarrow 0$ as $n \rightarrow \infty$. The converse does not hold in general; consider $f \equiv 1$. However, we can achieve a partial converse.

For $A \in \mathcal{M}$, we say $f \in L^1(A)$ if $\int_A |f| < \infty$.

\hookrightarrow **Proposition 2.27:** Given $A \in \mathcal{M}$ with $m(A) < \infty$, then

$$f \in L^1(A) \Leftrightarrow \lim_n \int_{A \cap \{|f|>n\}} |f| = 0.$$

PROOF. (\Rightarrow) We've proven before, c.f. properties of integral of non-negative functions.

(\Leftarrow) Choose N such that $\int_{A \cap \{|f|>N\}} |f| \leq 1$. Then,

$$\begin{aligned} \int_A |f| &= \int_{A \cap \{|f| \leq N\}} |f| + \int_{A \cap \{|f| > N\}} |f| \\ &\leq N \cdot m(A) + 1 < \infty. \end{aligned}$$

■

↪ **Definition 2.13** (Uniform Integrability): Given $\{f_n\}$ measurable and $A \in \mathcal{M}$, we say $\{f_n\}$ is uniformly integrable on A if

$$\lim_{M \rightarrow \infty} \left(\sup_{n \geq 1} \left(\int_{A \cap \{|f_n| > M\}} |f_n| \right) \right) = 0.$$

↪ **Proposition 2.28**: Let $\{f_n\}$ measurable, $A \in \mathcal{M}$.

1. If $m(A) < \infty$ and $\{f_n\}$ uniformly integrable on A , then $\{f_n\}$ is bounded in $L^1(A)$, that is $\sup_{n \geq 1} \int_A |f_n| < \infty$.
2. If $\{f_n\}$ is bounded in $L^p(A)$ for any $1 < p < \infty$, then $\{f_n\}$ is uniformly integrable on A .

PROOF.

1. Let M such that $\sup_{n \geq 1} \int_{A \cap \{|f_n| > M\}} |f_n| \leq 1$. Then, we have that

$$\begin{aligned} \sup_{n \geq 1} \int_A |f_n| &= \sup_{n \geq 1} \left(\int_{A \cap \{|f_n| \leq M\}} |f_n| + \int_{A \cap \{|f_n| > M\}} |f_n| \right) \\ &\leq M \cdot m(A) + 1 < \infty. \end{aligned}$$

2. For any $M > 0$, note that $1 \leq \left(\frac{|f_n|}{M}\right)^{p-1}$ over $A \cap \{|f_n| > M\}$. So,

$$\begin{aligned} \sup_n \int_{A \cap \{|f_n| > M\}} |f_n| &\leq \sup_n \int_{A \cap \{|f_n| > M\}} |f_n| \left(\frac{|f_n|}{M}\right)^{p-1} \\ &\leq \underbrace{\frac{1}{M^{p-1}}}_{>0} \underbrace{\sup_n \int_A |f_n|^p}_{< \infty} \rightarrow 0 \text{ as } M \rightarrow \infty. \end{aligned}$$

■

Remark 2.23: Notice that 2. does *not* require finiteness of the measure of A , in particular one can take $A = \mathbb{R}$.

↪ **Proposition 2.29**: Given $\{f_n\}$ measurable and $A \in \mathcal{M}$ with $m(A) < \infty$, TFAE:

- (i) $f_n \in L^1(A) \forall n \geq 1, f \in L^1(A)$ and $f_n \rightarrow f$ in $L^1(A)$,
- (ii) $\{f_n\}$ is uniformly integrable on A and $f_n \rightarrow f$ in measure on A .

PROOF. (i) \Rightarrow (ii) Assume $f_n \rightarrow f$ in $L^1(A)$, hence $\int_A |f_n| \rightarrow \int_A |f|$ so $\{f_n\}$ bounded in $L^1(A)$. Note we've already proven that $f_n \rightarrow f$ in measure. For $M > 0$,

$$\begin{aligned}
\int_{A \cap \{|f_n| > M\}} |f_n| &\leq \int_{A \cap \{|f_n| > M\}} |f_n - f| + \int_{A \cap \{|f_n| > M\}} |f| \\
&\leq \underbrace{\int_A |f_n - f|}_{\rightarrow 0} + \underbrace{\int_{A \cap \{|f_n| > M\} \cap \{|f| \leq \sqrt{M}\}} |f|}_{\leq \sqrt{M} \cdot m(A \cap \{|f_n| > M\})} + \underbrace{\int_{A \cap \{|f_n| > M\} \cap \{|f| > \sqrt{M}\}} |f|}_{\leq \int_{A \cap \{|f| > \sqrt{M}\}} |f| \rightarrow 0 \text{ since } f \in L^1} \\
&\leq \sqrt{M} \frac{\sup_n \int_A |f_n|}{M} \rightarrow 0 \text{ as } M \rightarrow \infty \\
&\quad \text{(Markov's)}
\end{aligned}$$

Fix $\varepsilon > 0$. Choose N such that for all $n \geq N$, $\int_A |f_n - f| \leq \frac{\varepsilon}{3}$, choose M such that $\int_{A \cap \{|f| > \sqrt{M}\}} |f| < \frac{\varepsilon}{3}$ and $\frac{\sup_n \int_A |f_n|}{\sqrt{M}} < \frac{\varepsilon}{3}$. Thus,

$$\sup_{n \geq N} \int_{A \cap \{|f_n| > M\}} |f_n| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

We want this to hold for $N = 1$ for uniformity, i.e. we need to deal with the first $N - 1$ terms. We achieve this by making M larger if necessary such that

$$\int_{A \cap \{|f_k| > M\}} |f_k| \leq \varepsilon$$

for every $k = 1, 2, \dots, N - 1$.

(ii) \Rightarrow (i) assignment question. ■

§3 PRODUCT SPACE

§3.1 Preparations

Given a measure space (X, \mathcal{F}, μ) with μ a σ -finite measure (i.e. there exists a sequence $\{X_n\} \subseteq \mathcal{F}$ such that $X_n \uparrow$ and $\bigcup_n X_n = X$, and $\mu(X_n) < \infty$ for each n).

\Leftrightarrow **Definition 3.1** (Measurable): $f : X \rightarrow \bar{\mathbb{R}}$ is \mathcal{F} -measurable if $\forall a \in \mathbb{R}, f^{-1}([-\infty, a)) \in \mathcal{F}$.

We have similar properties for f in general as in the Lebesgue setting. -For f \mathcal{F} -measurable, $cf, f^k, |f|, f \wedge a, f \vee b, f^+, f^-$ are all \mathcal{F} -measurable for $a, b, c \in \mathbb{R}$.

- For f, g \mathcal{F} -measurable, $f + g, f - g, f \cdot g, f \wedge g, f \vee g$ are all \mathcal{F} -measurable.
- If $\{f_n\}$ \mathcal{F} -measurable, $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \liminf_n f_n$ are \mathcal{F} -measurable.

We may "dissect" functions as before. For f \mathcal{F} -measurable, write $f = f^+ - f^-$, and put for $n \geq 1$ and $\bullet = +, -$,

$$f_n^\bullet := \mathbb{1}_{X_n}(f^\bullet \wedge n).$$

Then, $f_n^\bullet \uparrow f^\bullet$. Put

$$\varphi_n^\bullet := \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}^\bullet},$$

where, for $k = 0, 1, \dots, n2^n$ for $n \geq 1$,

$$A_{n,k}^\bullet = \left\{ x \in X_n : \frac{k}{2^n} \leq f_n^\bullet < \frac{k+1}{2^n} \right\} \in \mathcal{F}.$$

Then, we may define the integral of the simple function

$$\int_X \varphi_n^\bullet d\mu := \sum_{k=0}^{n2^n} \frac{k}{2^n} \mu(A_{n,k}^\bullet).$$

Define then

$$\int_X f^\bullet d\mu := \lim_n \int_X \varphi_n^\bullet d\mu,$$

and

$$\int_X f d\mu := \int_X f^+ d\mu - \int_X f^- d\mu.$$

We say, then, $f \in L^1(\mu)$ if $\int_X |f| d\mu < \infty$. This generalizes the notion of integration to a (slightly more) general σ -algebra.

§3.2 Product Lebesgue σ -Algebra

We will restrict our constructions to the product of 2 spaces, i.e. \mathbb{R}^2 , but generalizes for general \mathbb{R}^d .

↪ **Definition 3.2** (Product σ -algebra): The *product σ -algebra* of subsets of \mathbb{R}^2 , denoted by $\mathcal{M} \otimes \mathcal{M}$ or simply \mathcal{M}^2 , is defined as

$$\mathcal{M}^2 := \sigma(\{A \times B : A, B \in \mathcal{M}\}),$$

where

$$A \times B := \{(x, y) : x \in A, y \in B\}$$

as is standard.

Notice \mathcal{M}^2 contains

- rectangles $I_1 \times I_2$, I_1, I_2 intervals;
- singletons $\{(x, y)\}$;
- open sets, closed sets, and so $\mathfrak{B}(\mathbb{R}^2) := \sigma(\{\text{open sets in } \mathbb{R}^2\}) \subseteq \mathcal{M}^2$.

Given G open, then for every $x \in G$, there exists some disc centered at x contained entirely in G . Moreover, there exist $(a_1, a_2), (b_1, b_2)$ with $a_i, b_i \in \mathbb{Q}$ such that $x \in (a_1, a_2) \times (b_1, b_2) \subset G$. Then, $G = \bigcup_{x \in G} (a_1, a_2) \times (b_1, b_2)$.

↪ **Definition 3.3** (Slice): Given $E \subseteq \mathbb{R}^2$, then for every $x \in \mathbb{R}$, define

$$E_x := \{y \in \mathbb{R} : (x, y) \in E\} \subseteq \mathbb{R},$$

called the *slice* of E at x . Similarly, define for $y \in \mathbb{R}$,

$$E^y := \{x \in \mathbb{R} : (x, y) \in E\} \subseteq \mathbb{R}.$$

↪ **Proposition 3.1:** If $E \in \mathcal{M}^2$, then for every $x \in \mathbb{R}$, $E_x \in \mathcal{M}$, and for every $y \in \mathbb{R}$ $E^y \in \mathcal{M}$; that is, product measurability \Rightarrow marginal measurability.

PROOF. Define

$$\mathcal{A} := \{E \subseteq \mathbb{R}^2 : \forall x \in \mathbb{R}, E_x \in \mathcal{M}\}.$$

We claim \mathcal{A} a σ -algebra of subsets of \mathbb{R}^2 .

- $\mathbb{R}^2 \in \mathcal{A}$? Yes, since for every $x \in \mathbb{R}$, $\mathbb{R}_x^2 = \mathbb{R} \in \mathcal{M}$.
- Let $E \in \mathcal{A}$. Then, $E_x \in \mathcal{M}$ for every $x \in \mathbb{R}$. But we have too

$$(E^c)_x = (E_x)^c,$$

and since $E_x \in \mathcal{M} \Rightarrow (E_x)^c \in \mathcal{M}$, it follows that $E^c \in \mathcal{A}$.

- If $\{E_n\} \subseteq \mathcal{A}$, then for every $x \in \mathbb{R}$,

$$\left(\bigcup_n E_n\right)_x = \left(\bigcup_n (E_n)_x\right) \in \mathcal{M}$$

so $\bigcup_n E_n \in \mathcal{A}$.

Hence, \mathcal{A} indeed a σ -algebra of subsets of \mathbb{R}^2 . For every $A, B \in \mathcal{M}$, we claim $A \times B \in \mathcal{A}$. We have that for every $x \in \mathbb{R}$,

$$(A \times B)_x = \begin{cases} \emptyset & \text{if } x \notin A \\ B & \text{if } x \in A \end{cases} \in \mathcal{M},$$

hence $A \times B \in \mathcal{A}$. Thus, since such sets generate \mathcal{M}^2 , it follows that $\mathcal{M}^2 \subseteq \mathcal{A}$, and so every set in \mathcal{M}^2 has the desired property.

An identical proof follows for E^y -type slices. ■

Remark 3.1: Notice we didn't prove $\mathcal{A} = \mathcal{M}^2$, indeed, because its not true.

For instance, let $E = N \times A$ with N the Vitali set and $A \in \mathcal{M}$. Then, for every $x \in A$, $E_x = \begin{cases} A & \text{if } x \in N \\ \emptyset & \text{if } x \notin N \end{cases} \in \mathcal{M}$, but $E \notin \mathcal{M}^2$, because for every $y \in \mathbb{R}$, $E^y = \begin{cases} N & \text{if } y \in A \\ \emptyset & \text{else} \end{cases}$.

In fact, there exists sets such that E_x and $E^y \in \mathcal{M}$ for every $x, y \in \mathbb{R}$, but $E \notin \mathcal{M}^2$ (the *Sierpinski set*).

However, if $E \subseteq \mathbb{R}^2$ a product set, i.e. $E = A \times B$ for some $A, B \subseteq \mathbb{R}$, then $A, B \in \mathcal{M} \Rightarrow E \in \mathcal{M}^2$.

↪ **Definition 3.4** (Slice of sets): Let $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$ a function. For every $x \in \mathbb{R}$, define

$$f_x : \mathbb{R} \rightarrow \overline{\mathbb{R}}, \quad f_x(y) := f(x, y),$$

called the *slice* of f at x . Similarly define f^y .

⊗ **Example 3.1:** If $f = \mathbb{1}_E$ for some $E \subseteq \mathbb{R}^2$, then $f_x = \mathbb{1}_{E_x}$.

↔ **Proposition 3.2:** If $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$ is \mathcal{M}^2 -measurable, then for every $x \in \mathbb{R}$, f_x is \mathcal{M} -measurable, and for every $y \in \mathbb{R}$ f^y is \mathcal{M} -measurable.

PROOF. Observe that for every $B \subseteq \mathbb{R}$,

$$(f^{-1}(B))_x = f_x^{-1}(B)$$

for every $x \in \mathbb{R}$, with similar for f^y . In particular, then, if f \mathcal{M}^2 -measurable, then for every $a \in \mathbb{R}$, $f^{-1}([-\infty, a)) \in \mathcal{M}^2$ hence $f_x^{-1}([-\infty, a)) = (f^{-1}([-\infty, a)))_x \in \mathcal{M}$, with the same idea following for f^y . ■

Remark 3.2:

- If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous, then f is measurable. For every $a \in \mathbb{R}$, $f^{-1}((-\infty, a))$ open by virtue (indeed, definition) of continuity, hence in \mathcal{M}^2 .
- If $f = \mathbb{1}_E$ for some $E \subseteq \mathbb{R}^2$, f \mathcal{M}^2 -measurable $\Leftrightarrow E \in \mathcal{M}^2$.
- In general, there exists $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$ such that f_x \mathcal{M} -measurable but f is not \mathcal{M}^2 -measurable.
- If $f(x, y) = h(x)g(y)$ for some non-trivial $h, g : \mathbb{R} \rightarrow \overline{\mathbb{R}}$, then f is \mathcal{M}^2 -measurable \Leftrightarrow both h and g are \mathcal{M} -measurable. We show \Leftarrow ;

$$\begin{aligned} f^{-1}([-\infty, a)) &= \{(x, y) : h(x)g(y) < a\} \\ &= \{(x, y) : h(x) = 0, 0 < a\} \\ &\quad \cup \left\{ (x, y) : h(x) > 0, g(y) < \frac{a}{h(x)} \right\} \\ &\quad \cup \left\{ (x, y) : h(x) < 0, g(y) > \frac{a}{h(x)} \right\} \\ &= \{x : h(x) = 0\} \times \mathbb{R} \cap \{0 < a\} \in \mathcal{M}^2 \\ &\quad \cup \left(\underbrace{\bigcup_{q \in \mathbb{Q}} \left\{ x : 0 < h(x), q < \frac{a}{h(x)} \right\}}_{\in \mathcal{M}} \times \underbrace{\{y : g(y) < q\}}_{\in \mathcal{M}} \right) \\ &\quad \cup \left(\underbrace{\bigcup_{q \in \mathbb{Q}} \left\{ x : 0 > h(x), q > \frac{a}{h(x)} \right\}}_{\in \mathcal{M}} \times \underbrace{\{y : g(y) > q\}}_{\in \mathcal{M}} \right) \in \mathcal{M}^2 \end{aligned}$$

§3.3 Product Measure

↪ **Definition 3.5:** Given $E \in \mathcal{M}^2$, define functions

$$I_E^{(1)} : \mathbb{R} \rightarrow \overline{\mathbb{R}}, \quad I_E^{(1)}(x) := m(E_x)$$

and

$$I_E^{(2)} : \mathbb{R} \rightarrow \overline{\mathbb{R}}, \quad I_E^{(2)}(y) := m(E^y).$$

↪ **Theorem 3.1:** Given $E \in \mathcal{M}^2$, $I_E^{(1)}, I_E^{(2)}$ are \mathcal{M} -measurable functions, and in particular

$$\int_{\mathbb{R}} I_E^{(1)}(x) dx = \int_{\mathbb{R}} I_E^{(2)}(y) dy. \quad \circledast$$

PROOF. If indeed $I_E^{(1)}, I_E^{(2)}$ \mathcal{M} -measurable, then the integrals of the functions are well-defined, being non-negative functions.

Set

$$\Sigma := \left\{ E \in \mathcal{M}^2 : I_{E_N}^{(1)}, I_{E_N}^{(2)} \text{ are } \mathcal{M}\text{-measurable and } \circledast \text{ holds, for } E_N := E \cap [-N, N]^2 \text{ for all } N > 0 \right\}.$$

Note that for every $E \in \mathcal{M}^2$, for all $N > 0$,

$$I_{E_N}^{(1)}(x) = \begin{cases} m((E_N)_x) & \text{if } x \in [-N, N] \\ 0 & \text{o.w.} \end{cases} = \mathbb{1}_{[-N, N]}(x) I_{E_N}^{(1)}(x).$$

similarly for $I_{E_N}^{(2)}$.

Let $\mathcal{C} := \{A \times B : A, B \in \mathcal{M}\}$ (recall $\mathcal{M}^2 = \sigma(\mathcal{C})$).

- *Claim 1:* $\mathcal{C} \subseteq \Sigma$

For every $N > 0$, $E_N = (A \times B) \cap [-N, N]^2 = A_N \times B_N$ ($A_N := A \cap [-N, N]$). Then,

$$I_{E_N}^{(1)}(x) = I_{A_N \times B_N}^{(1)}(x) = \begin{cases} m(B_N) & \text{if } x \in A_N \\ 0 & \text{if } x \notin A_N \end{cases}, \quad I_{E_N}^{(2)}(y) = I_{A_N \times B_N}^{(2)}(y) = \begin{cases} m(A_N) & \text{if } y \in B_N \\ 0 & \text{if } y \notin B_N \end{cases}$$

and so $I_{E_N}^{(1)}, I_{E_N}^{(2)}$ are measurable seeing as they are both just indicator functions of measurable sets times a constant. In particular,

$$\int_{\mathbb{R}} I_{E_N}^{(1)} = m(B_N)m(A_N) = \int_{\mathbb{R}} I_{E_N}^{(2)}$$

as required. Hence, indeed $E_N \in \Sigma$ and so $\mathcal{C} \subseteq \Sigma$.

- *Claim 2:* $\mathbb{R}^2 \in \Sigma$

For every $N > 0$,

$$I_{[-N, N]^2}^{(1)}(x) = \begin{cases} 2N & \text{if } x \in [-N, N] \\ 0 & \text{o.w.} \end{cases},$$

similar for $I_{[-N, N]^2}^{(2)}, I_{[-N, N]}^{(1)}, I_{[-N, N]}^{(2)}$ are both \mathcal{M} -measurable, and their integrals agree, and so it follows that $\mathbb{R}^2 \in \Sigma$.

- *Claim 3:* $E \in \Sigma \Rightarrow E^c \in \Sigma$

For each $N > 0$, denote

$$F_N := E^c \cap [-N, N]^2.$$

$I_{F_N}^{(1)} = 0$ outside of $[-N, N]$, and for $x \in [-N, N]$,

$$(F_N)_x = \{y : (x, y) \in E^c \cap [-N, N]^2\} = [-N, N] \setminus E_x = [-N, N] \setminus (E_N)_x$$

so

$$I_{F_N}^{(1)}(x) = 2N - I_{E_N}^{(1)}(x)$$

for $x \in [-N, N]$. Similarly,

$$I_{F_N}^{(2)}(y) = \begin{cases} 2N - I_{E_N}^{(2)}(y) & \text{if } y \in [-N, N] \\ 0 & \text{o.w.} \end{cases}$$

In particular, then, $I_{F_N}^{(1)}, I_{F_N}^{(2)}$ measurable, and

$$\begin{aligned} \int_{\mathbb{R}} I_{F_N}^{(1)} &= \int_{[-N, N]} 2N - I_{E_N}^{(1)} = 4N^2 - \int_{\mathbb{R}} I_{E_N}^{(1)} \\ \int_{\mathbb{R}} I_{F_N}^{(2)} &= \int_{[-N, N]} 2N - I_{E_N}^{(2)} = 4N^2 - \int_{\mathbb{R}} I_{E_N}^{(2)}, \end{aligned}$$

but we know $\int_{\mathbb{R}} I_{E_N}^{(1)} = \int_{\mathbb{R}} I_{E_N}^{(2)}$ since $E_N \in \Sigma$, hence it follows that $\int_{\mathbb{R}} I_{F_N}^{(1)} = \int_{\mathbb{R}} I_{F_N}^{(2)}$ and so it follows that $E^c \in \Sigma$.

- *Claim 4:* $\{E_k\} \subseteq \Sigma \Rightarrow E := \bigcup_{k=1}^{\infty} E_k \in \Sigma$.

Wlog, E_n 's disjoint. For $N > 0$, $E_N = \bigcup_{k=1}^{\infty} E_{k,N}$.

$$I_{E_N}^{(1)}(x) = \mathbb{1}_{[-N, N]}(x) m\left(\bigcup_{k=1}^{\infty} (E_{k,N})_x\right) = \mathbb{1}_{[-N, N]} \sum_{k=1}^{\infty} m((E_{k,N})_x) = \sum_{k=1}^{\infty} I_{E_{k,N}}^{(1)}(x),$$

with similarly $I_{E_N}^{(2)}(y) = \sum_{k=1}^{\infty} I_{E_{k,N}}^{(2)}(y)$. This implies $I_{E_N}^{(1)}, I_{E_N}^{(2)}$ are \mathcal{M} -measurable, and in particular

$$\int_{\mathbb{R}} I_{E_N}^{(1)} = \sum_{k=1}^{\infty} \int_{\mathbb{R}} I_{E_{k,N}}^{(1)}, \quad \int_{\mathbb{R}} I_{E_N}^{(2)} = \sum_{k=1}^{\infty} \int_{\mathbb{R}} I_{E_{k,N}}^{(2)},$$

which are equal since by assumption $E_k \in \Sigma$. Hence, $E \in \Sigma$, and thus by Claims 2-4, Σ a σ -algebra of subsets of \mathbb{R}^2 , and thus by Claim 1 $\Sigma = \mathcal{M}^2$.

Hence, for every $E \in \mathcal{M}^2$, $E \in \Sigma$ and so all the statements hold for E_N for every $N > 0$. Then,

$$I_E^{(1)}(x) = \lim_{N \rightarrow \infty} \mathbb{1}_{[-N, N]}(x) m((E_N)_x) = \lim_{N \rightarrow \infty} m((E_N)_x) = m(E_x) = \lim_{N \rightarrow \infty} I_{E_N}^{(1)}(x),$$

and in particular $\{I_{E_N}^{(1)}\} \uparrow$, hence $I_E^{(1)}$ \mathcal{M} -measurable, and

$$\int_{\mathbb{R}} I_E^{(1)} = \lim_{N \rightarrow \infty} \int_{\mathbb{R}} I_{E_N}^{(1)},$$

with similar for $I_E^{(2)}$, by monotonicity. Thus, since $\int_{\mathbb{R}} I_{E_N}^{(1)} = \int_{\mathbb{R}} I_{E_N}^{(2)}$ for every N , the proof follows. ■

↪ **Definition 3.6:** Define a non-negative set function on $(\mathbb{R}^2, \mathcal{M}^2)$ by

$$m(E) := \int_{\mathbb{R}} I_E^{(1)}(x) dx = \int_{\mathbb{R}} I_E^{(2)}(x) dx, \quad E \in \mathcal{M}^2.$$

m is called the *Lebesgue measure on \mathbb{R}^2* .

↪ **Proposition 3.3:** m is indeed a measure on $(\mathbb{R}^2, \mathcal{M}^2)$.

PROOF.

- $m(\emptyset) = \int_{\mathbb{R}} 0 = 0$
- If $\{E_k\} \subseteq \mathcal{M}^2$ disjoint, let $E = \bigcup_{k=1}^{\infty} E_k$. Then

$$m(E) = \sum_{k=1}^{\infty} m(E_k),$$

since for every $x \in \mathbb{R}$, $E_x = \bigcup_{k=1}^{\infty} (E_k)_x$ disjoint, so

$$\int_{\mathbb{R}} I_E^{(1)} = \sum_{k=1}^{\infty} \int_{\mathbb{R}} I_{E_k}^{(1)},$$

and the proof follows. ■

Remark 3.3:

1. For any $E = I_1 \times I_2$, $m(E) = \ell(I_1) \cdot \ell(I_2)$. It follows that any singleton, and countable set, and any line on \mathbb{R}^2 is a null set.
2. If $A \subseteq \mathbb{R}$ is a null set in \mathcal{M} , then $A \times \mathbb{R}, \mathbb{R} \times A$ are null sets, in \mathcal{M}^2 .
3. \mathcal{M}^2 is *not* complete under m , since for instance if $N \subset \mathbb{R}$ the Vitali set, $a \in \mathbb{R}$, then $\{a\} \times N \subseteq \{a\} \times \mathbb{R}$ is a subset of a null set, but $\{a\} \times N$ is not measurable.
4. It is possible to construct m on \mathbb{R}^2 through the “outer measure” approach. We take $E \subseteq \mathbb{R}^2$, and define

$$m^*(E) = \inf \left\{ \sum_{n=1}^{\infty} \text{Area}(R_n) : R_n \text{ 's closed, finite rectangles s.t. } E \subseteq \bigcup_{n=1}^{\infty} R_n \right\}.$$

Then, m^* satisfies similar properties as the 1-dimensional analog. We then say a set E is measurable if for every $F \subseteq \mathbb{R}^2$, $m^*(F) = m^*(F \cap E) + m^*(F \cap E^c)$. Collect all such sets, $\overline{\mathcal{M}^2} := \{E \subseteq \mathbb{R}^2 : E \text{ measurable}\}$. This is a σ -algebra of subsets of \mathbb{R}^2 , with $m := m^*|_{\overline{\mathcal{M}^2}}$ a measure when restricted to it. Indeed, m matches the Lebesgue measure defined above, and $\overline{\mathcal{M}^2}$, as suggestively notated, the completion of \mathcal{M}^2 under the Lebesgue measure. In addition, $\overline{\mathcal{M}^2} = \mathfrak{B}_{\mathbb{R}^2}$.

5. The Lebesgue measure m on \mathbb{R}^2 is the unique measure on $\mathcal{M}^2/\mathfrak{B}_{\mathbb{R}^2}/\overline{\mathcal{M}^2}$ such that for all $I_1 \times I_2$ rectangles, $m(I_1 \times I_2) = \ell(I_1)\ell(I_2)$. This is because $\mathcal{I} := \{I_1 \times I_2 : I_1, I_2 \text{ finite intervals}\}$ is a π -system and generates $\mathfrak{B}_{\mathbb{R}^2}$.
6. The Lebesgue measure on \mathbb{R}^2 is translation invariant (rectangle area is invariant under translation). Namely, show that $m_z : \mathcal{M}^2 \rightarrow [0, \infty]$, $m_z(E) := m(E + z)$ is a measure and $m_z = m$ on \mathcal{I} .
7. The Lebesgue measure m on \mathbb{R}^2 is the only measure on $\mathcal{M}^2/\mathfrak{B}_{\mathbb{R}^2}/\overline{\mathcal{M}^2}$ that is translation invariant and assigns 1 to $[0, 1] \times [0, 1]$.

§3.4 Fubini’s Theorem

↔ **Definition 3.7:** Let $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$ be \mathcal{M}^2 -measurable and non-negative. Define the functions

$$I_f^{(1)}(x) := \int_{\mathbb{R}} f(x, y) dy = \int_{\mathbb{R}} f_x(y) dy, \quad I_f^{(2)}(y) := \int_{\mathbb{R}} f(x, y) dx = \int_{\mathbb{R}} f^y(x) dx.$$

Remark 3.4: Given $f : \mathbb{R}^2 \rightarrow [0, \infty]$, \mathcal{M}^2 -measurable and non-negative, the integral of f wrt the Lebesgue measure on \mathbb{R}^2 is denoted by $\int_{\mathbb{R}^2} f(x, y) dx dy$ or $\int_{\mathbb{R}^2} f$ if there is no ambiguity.

↪ **Theorem 3.2** (Tonelli's): Let $f : \mathbb{R}^2 \rightarrow [0, \infty]$ be \mathcal{M}^2 -measurable and non-negative. Then,

$$\int_{\mathbb{R}^2} f(x, y) \, dx \, dy = \int_{\mathbb{R}} I_f^{(1)}(x) \, dx = \int_{\mathbb{R}} I_f^{(2)}(y) \, dy,$$

or more explicitly,

$$\int_{\mathbb{R}^2} f(x, y) \, dx \, dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y) \, dy \right) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x, y) \, dx \right) dy.$$

PROOF. Since f \mathcal{M}^2 -measurable, non-negative, there exists $\{\varphi_n\}$ -sequence of simple functions with $\varphi_n \uparrow f$, and $\int_{\mathbb{R}^2} f = \lim_n \int_{\mathbb{R}^2} \varphi_n$, where, eg

$$\varphi_n = \sum_{k=0}^{n2^n} \frac{k}{2^n} \mathbb{1}_{A_{n,k}}, \quad A_{n,k} := \left\{ (x, y) \in [-n, n]^2 : \frac{k}{2^n} \leq f(x, y) < \frac{k+1}{2^n} \right\}, \quad k = 0, 1, \dots, n2^n.$$

So,

$$\int_{\mathbb{R}^2} f(x, y) \, dx \, dy = \lim_n \sum_{k=0}^{n2^n} \frac{k}{2^n} m(A_{n,k}).$$

On the other hand, $\forall x \in \mathbb{R}$, by MON

$$\begin{aligned} I_f^{(1)}(x) &= \int_{\mathbb{R}} f(x, y) \, dy = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \varphi_n(x, y) \, dy \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} I_{A_{n,k}}^{(1)}(x) \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} m((A_{n,k})_x). \end{aligned}$$

We have then, again by MON, that

$$\int_{\mathbb{R}} I_f^{(1)}(x) \, dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \sum_{k=0}^{n2^n} \frac{k}{2^n} I_{A_{n,k}}^{(1)}(x) \, dx = \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} \int_{\mathbb{R}} I_{A_{n,k}}^{(1)}(x) \, dx.$$

Similarly, we find

$$\int_{\mathbb{R}} I_f^{(2)}(y) \, dy = \lim_{n \rightarrow \infty} \sum_{k=0}^{n2^n} \frac{k}{2^n} \int_{\mathbb{R}} I_{A_{n,k}}^{(2)}(y) \, dy.$$

By definition,

$$m(A_{n,k}) = \int_{\mathbb{R}} I_{A_{n,k}}^{(1)}(x) \, dx = \int_{\mathbb{R}} I_{A_{n,k}}^{(2)}(y) \, dy,$$

hence all of our terms actually agree, and bringing them together gives the proof. ■

↪ **Definition 3.8:** Given $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$ \mathcal{M}^2 -measurable, we write $f \in L^1(\mathbb{R}^2)$ if

$$\int_{\mathbb{R}^2} |f(x, y)| \, dx \, dy < \infty,$$

or equivalently if

$$\int_{\mathbb{R}^2} f^+ \text{ and } \int_{\mathbb{R}^2} f^- < \infty.$$

Remark 3.5: Suppose $f \in L^1(\mathbb{R}^2)$. Then by Tonelli's,

$$\int_{\mathbb{R}^2} |f(x, y)| \, dx \, dy = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x, y)| \, dy \right) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x, y)| \, dx \right) dy,$$

and in particular all integrals are finite; namely, $I_{|f|}^{(1)}, I_{|f|}^{(2)} \in L^1(\mathbb{R})$.

↪ **Theorem 3.3** (Fubini's): If $f : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}$ is \mathcal{M}^2 -measurable and $f \in L^1(\mathbb{R}^2)$, then

1. $I_f^{(1)}, I_f^{(2)} \in L^1(\mathbb{R})$ (product integrability \Rightarrow marginal integrability)
2. $I_f^{(1)}(x)$ finite-valued for a.e. $x \in \mathbb{R} \Rightarrow f_x \in L^1(\mathbb{R})$ for a.e. $x \in \mathbb{R}$, similar for $I_f^{(2)}$, i.e. $f^y \in L^1(\mathbb{R})$ for a.e. $y \in \mathbb{R}$.
3. $\int_{\mathbb{R}^2} f(x, y) \, dx \, dy = \int_{\mathbb{R}} I_f^{(1)}(x) \, dx = \int_{\mathbb{R}} I_f^{(2)}(y) \, dy$.

PROOF. Assume $f \in L^1(\mathbb{R}^2)$. Then by Tonelli's

$$\int_{\mathbb{R}} |I_f^{(1)}| \leq \int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(x, y)| \, dy \right) dx = \int_{\mathbb{R}^2} |f(x, y)| \, dx \, dy < \infty \Rightarrow I_f^{(1)} \in L^1(\mathbb{R}).$$

We have 1. \Rightarrow 2..

Now, write $f = f^+ - f^-$. $f \in L^1(\mathbb{R}^2)$ gives that $f^+, f^- \in L^1(\mathbb{R}^2)$ so f^+, f^- each finite valued a.e.. By Tonelli's, then,

$$\int_{\mathbb{R}} I_{f^+}^{(1)}(x) \, dx = \int_{\mathbb{R}} I_{f^+}^{(2)}(y) \, dy = \int_{\mathbb{R}^2} f^+(x, y) \, dx \, dy < \infty,$$

same with f^- . Then, $I_{f^+}^{(1)}, I_{f^+}^{(2)}, I_{f^-}^{(1)}, I_{f^-}^{(2)} \in L^1(\mathbb{R})$, hence are finite-valued a.e.. By linearity on L^1 functions, then

$$\int_{\mathbb{R}} I_{f^+}^{(1)}(x) \, dx - \int_{\mathbb{R}} I_{f^-}^{(1)}(x) \, dx = \int_{\mathbb{R}} I_{f^+}^{(1)} - I_{f^-}^{(1)}.$$

For a.e. $x \in \mathbb{R}$, $f_x^+, f_x^- \in L^1(\mathbb{R})$, so by linearity

$$I_{f^+}^{(1)}(x) - I_{f^-}^{(1)}(x) = \int_{\mathbb{R}} f_x^+(y) \, dy - \int_{\mathbb{R}} f_x^-(y) \, dy = \int_{\mathbb{R}} (f_x^+ - f_x^-) = \int_{\mathbb{R}} f_x,$$

so

$$\int_{\mathbb{R}} I_{f^+}^{(1)} - \int_{\mathbb{R}} I_{f^-}^{(1)} = \int_{\mathbb{R}} I_f^{(1)},$$

with similarly for

$$\int_{\mathbb{R}} I_{f^+}^{(2)} - \int_{\mathbb{R}} I_{f^-}^{(2)} = \int_{\mathbb{R}} I_f^{(2)}.$$

All together, then,

$$\int_{\mathbb{R}} I_f^{(1)} = \int_{\mathbb{R}} I_f^{(2)} = \int_{\mathbb{R}^2} (f^+ - f^-) = \int_{\mathbb{R}^2} f.$$

■

Remark 3.6: In general, $I_f^{(1)}, I_f^{(2)} \in L^1(\mathbb{R}) \not\Rightarrow f \in L^1(\mathbb{R}^2)$. For instance, let

$$f(x, y) = \begin{cases} 1 & \text{if } x < y < x + 1 \\ -1 & \text{if } x - 1 < y < x \\ 0 & \text{else} \end{cases}$$

Then, for all $x \in \mathbb{R}$,

$$I_f^{(1)}(x) = \int_{\mathbb{R}} f(x, y) dy = 0,$$

and similarly

$$I_f^{(2)}(y) = \int_{\mathbb{R}} f(x, y) dx = 0,$$

so $I_f^{(1)}, I_f^{(2)} \in L^1(\mathbb{R})$, but $f \notin L^1(\mathbb{R}^2)$.

Remark 3.7: If $f : \mathbb{R}^2 \rightarrow \bar{\mathbb{R}}$ is $\overline{\mathcal{M}^2}$ -measurable, Tonelli's, Fubini's still hold. We'll use the same notations in this case.

In fact, there exists a $\tilde{f} : \mathbb{R}^2 \rightarrow \bar{\mathbb{R}}$ that is \mathcal{M}^2 -measurable such that $\tilde{f} = f$ a.e. (exercise).

Remark 3.8: The constructions above extend to \mathbb{R}^d , $d \geq 3$. In particular, we have

- $\mathcal{M}^d := \sigma(\{A_1 \times \dots \times A_d : A_i \in \mathcal{M}\})$.
- The product measure m is the Lebesgue measure on $(\mathbb{R}^d, \mathcal{M}^d)$.
- $\overline{\mathcal{M}^d}$ is the completion of \mathcal{M}^d under m .
- Tonelli's, Fubini's hold, with "d-embedded" integrals.
- m shares similar properties on \mathbb{R}^d as on \mathbb{R} ;
 - translation invariance,
 - scaling property,
 - regularity, (outer: for every $E \in \mathcal{M}^d$, $m(E) = \inf\{m(G) : G \text{ open s.t. } E \subseteq G\}$, inner: for every $E \in \mathcal{M}^d$, $m(E) = \sup\{m(K) : K \text{ compact s.t. } E \supseteq K\}$).

§4 DIFFERENTIATION

In the Riemann setting, differentiation and integration are closely related. For instance, if $F(x) := \int_a^x f(t) dt$ for some Riemann-integrable f on $[a, b]$ and $x \in [a, b]$, then F is differentiable and $F' = f$ on (a, b) . Or, if F differentiable, and F' is Riemann integrable on some $[a, b]$, then $F(b) - F(a) = \int_a^b F'(t) dt$. How much does this extend to the Lebesgue setting?

§4.1 Hardy-Littlewood Maximal Function

↪ **Definition 4.1** (Hardy-Littlewood Maximal Function): Suppose $f \in L^1(\mathbb{R})$. The *Hardy-Littlewood Maximal Function* (H-L max.), denoted f^* , is defined as

$$f^*(x) := \sup_{I \in \mathcal{I}(x)} \frac{1}{m(I)} \int_I |f|,$$

where $\mathcal{I}(x) := \{I : I \text{ an open interval containing } x\}$.

↪ **Proposition 4.1:** Given $f \in L^1(\mathbb{R})$, f^* is measurable.

PROOF. $f^* \geq 0$, so it suffices to show that for every $a \geq 0$, $\{f^* > a\}$ is measurable. Let $x \in \{f^* > a\}$. Then, $a < f^*(x)$, hence there must exist some $I \in \mathcal{I}(x)$ such that $\frac{1}{m(I)} \int_I |f| > a$. I is open, and $x \in I$, so there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq I$. For every $y \in (x - \delta, x + \delta)$, $y \in I$, hence $I \in \mathcal{I}(y)$. So, $f^*(y) \geq \frac{1}{m(I)} \int_I |f| > a$. Thus, $y \in \{f^* > a\}$ as well. It follows, then, that $(x - \delta, x + \delta) \subseteq \{f^* > a\}$, hence $\{f^* > a\}$ is open, and so in particular is measurable. ■

↪ **Lemma 4.1** (Vitali's Covering Lemma): Assume that $\mathcal{I} := \{I_1, \dots, I_N\}$ a finite collection of open intervals. Then, there exists a sub-collection $\{I_{k_1}, \dots, I_{k_M}\} \subset \mathcal{I}$ such that $I_{k_i} \cap I_{k_j} = \emptyset$ for all $i \neq j$ and

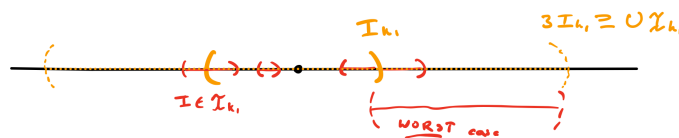
$$m\left(\bigcup_{i=1}^N I_i\right) \leq 3 \sum_{j=1}^M m(I_{k_j}).$$

PROOF. Assume wlog that $m(I_i) < \infty$ for all $1 \leq i \leq N$; if otherwise exists an i such that $m(I_i) = \infty$, then simply take your subcollection $I_{k_1} := I_i$, and the claim holds trivially.

Begin with the largest interval in \mathcal{I} , call it I_{k_1} . Let

$$\mathcal{J}_{k_1} := \{I \in \mathcal{I} : I \cap I_{k_1} \neq \emptyset\}.$$

For any $I \in \mathcal{J}_{k_1}$, $I \cap I_{k_1} \neq \emptyset$ and $m(I) \leq m(I_{k_1})$, so in particular $I \subseteq 3I_{k_1}$ (if $I_{k_1} = (a, b)$, $3I_{k_1} := (a - 3(b - a), a + 3(b - a))$).



Then, in particular

$$\bigcup_{I \in \mathcal{J}_{k_1}} I \subseteq 3I_{k_1}.$$

Consider now $\mathcal{I} \setminus \mathcal{J}_{k_1}$, and choose the largest interval in the remaining part of the collection. Call it I_{k_2} . Set

$$\mathcal{J}_{k_2} := \{I \in \mathcal{I} \setminus \mathcal{J}_{k_1} : I \cap I_{k_2} \neq \emptyset\}.$$

Similarly to before, $\bigcup_{I \in \mathcal{J}_{k_2}} I \subseteq 3I_{k_2}$. By choice, too, $I_{k_1} \cap I_{k_2} = \emptyset$.

Repeat this process, until $\mathcal{I} \setminus (\mathcal{J}_{k_1} \cup \dots \cup \mathcal{J}_{k_M}) = \emptyset$, i.e. we have no intervals left in the original collection. Then, we obtain I_{k_1}, \dots, I_{k_M} disjoint intervals, with corresponding subcollections $\mathcal{J}_{k_1}, \dots, \mathcal{J}_{k_M}$ forming a partition of \mathcal{I} . Then,

$$m\left(\bigcup_{n=1}^N I_n\right) = \sum_{j=1}^M m\left(\bigcup_{I \in \mathcal{J}_{k_j}} I\right) \leq 3 \sum_{j=1}^M m(I_{k_j}).$$

■

↪ **Proposition 4.2:** Suppose $f \in L^1(\mathbb{R})$ and let f^* be the H-L max function of f . Then, for every $\varepsilon > 0$,

$$m(\{x \in \mathbb{R} : f^*(x) > \varepsilon\}) \leq \frac{3}{\varepsilon} \|f\|_1 = \frac{3}{\varepsilon} \int_{\mathbb{R}} |f|.$$

PROOF. Fix $\varepsilon > 0$ and put $B := \{x \in \mathbb{R} : f^*(x) > \varepsilon\}$. By inner regularity,

$$m(B) = \sup\{m(K) : B \supseteq K \text{ compact}\}.$$

It suffices to show then that $m(K) \leq \frac{3}{\varepsilon} \|f\|_1$ for every compact $K \subseteq B$. For every $x \in K$, $f^*(x) > \varepsilon$ so there exists some open interval I_x such that $x \in I_x$ and $\frac{1}{m(I_x)} \int_{I_x} |f| > \varepsilon$.

Hence, we may cover $K \subseteq \bigcup_{x \in K} I_x$. Since K compact it admits a finite subinterval, call it $\mathcal{I} = \{I_1, \dots, I_N\}$, such that $K \subseteq \bigcup_{n=1}^N I_n$. By the Covering Lemma,

$$m(K) \leq m\left(\bigcup_{n=1}^N I_n\right) \leq 3 \sum_{j=1}^M m(I_{k_j}),$$

for some disjoint subcollection I_{k_1}, \dots, I_{k_M} . Meanwhile, for every $1 \leq j \leq M$,

$$m(I_{k_j}) < \frac{1}{\varepsilon} \int_{I_{k_j}} |f|,$$

hence, we find

$$m(K) \leq 3 \sum_{j=1}^M \frac{1}{\varepsilon} \int_{I_{k_j}} |f| = \frac{3}{\varepsilon} \int_{\bigcup_{j=1}^M I_{k_j}} |f| \leq \frac{3}{\varepsilon} \int_{\mathbb{R}} |f| = \frac{3}{\varepsilon} \|f\|_1.$$

■

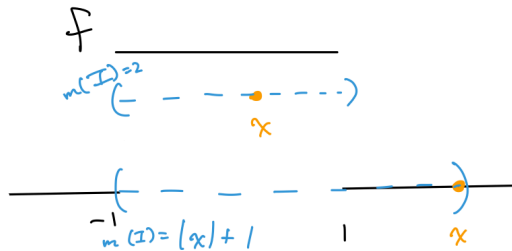
↪ **Corollary 4.1:** Given $f \in L^1(\mathbb{R})$, f^* is finite-valued a.e..

PROOF. For every $N > 0$, $m(\{f^* > N\}) \leq \frac{3}{N} \|f\|_1$. Taking $N \rightarrow \infty$, we find then $m(\{f^* > N\}) \rightarrow 0$, and since $m(\{f^* = \infty\}) \leq m(\{f^* > N\}) \forall N > 0$ it follows that $m(\{f^* = \infty\}) = 0$. ■

Remark 4.1: While a Markov-like inequality, f^* need not be integrable in general. For instance, let $f = \mathbb{1}_{[-1,1]} \in L^1(\mathbb{R})$. Then, consider f^* , and in particular consider the average of f over intervals $I = (a, b)$;

$$\frac{1}{b-a} \int_a^b f(t) dt = \begin{cases} 0 & \text{if } (a, b) \cap [-1, 1] = \emptyset \\ \frac{\min\{b, 1\} - \max\{a, -1\}}{b-a} & \text{if } (a, b) \cap [-1, 1] \neq \emptyset. \end{cases}$$

So, we find that $f^*(x) = 1$ if $x \in (-1, 1)$ (take your $I = [-1, 1]$, this achieves max), and $f^*(x) = \frac{2}{|x|+1}$ if $x \notin (-1, 1)$ (you want as much of the $[-1, 1]$ support as possible, and with your other endpoint as close to x as possible). f^* not integrable.



§4.2 Lebesgue Differentiation Theorem

↔ **Theorem 4.1** (Lebesgue Differentiation Theorem): Given $f \in L^1(\mathbb{R})$, for a.e. $x \in \mathbb{R}$, if $\{I_n\}$ a sequence of open intervals such that $x \in I_n \forall n \geq 1$ and $\lim_{n \rightarrow \infty} m(I_n) = 0$ (we say $\{I_n\}$ a sequence of intervals *shrinking to x*), then

$$\lim_{n \rightarrow \infty} \frac{1}{m(I_n)} \int_{I_n} |f(t) - f(x)| dt = 0.$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{1}{m(I_n)} \int_{I_n} f(t) dt = f(x).$$

PROOF. The “In particular” comes from the fact that, for x such that $f(x) < \infty$,

$$\left| \frac{1}{m(I_n)} \int_{I_n} f(t) dt - f(x) \right| = \left| \frac{1}{m(I_n)} \int_{I_n} f(t) - f(x) dt \right| \leq \frac{1}{m(I_n)} \int_{I_n} |f(t) - f(x)| dt,$$

so if the RHS $\rightarrow 0$, so does the left.

Without loss of generality, assume f finite valued everywhere, and only use finite-valued intervals I_n . For every $k \geq 1$, define

$$B_k := \left\{ x \in \mathbb{R} : \exists \{I_n\} \subseteq \mathcal{I}(x) \text{ with } \lim_{n \rightarrow \infty} m(I_n) = 0 \text{ s.t. } \limsup_{n \rightarrow \infty} \frac{1}{m(I_n)} \int_{I_n} |f(t) - f(x)| dt \geq \frac{1}{k} \right\}.$$

Notice $B_k \uparrow$ and $\bigcup_{k=1}^{\infty} B_k = \{x \in \mathbb{R} : \text{theorem fails}\}$. So, it suffices to show that $m(B_k) = 0$ for every $k \geq 1$.

Fix an arbitrary $\varepsilon > 0$. Continuous, compactly supported functions are dense in $L^1(\mathbb{R})$ so we may find $g \in C_c(\mathbb{R})$ such that $\|f - g\|_1 \leq \varepsilon$. Since g continuous and compactly supported, for every $x \in \mathbb{R}$ and $k \geq 1$, there exists some $\alpha > 0$ such that if $|t - x| \leq \alpha$, $|g(t) - g(x)| \leq \frac{1}{3k}$.

Given any $x \in \mathbb{R}$ and any sequence $\{I_n\} \subseteq \mathcal{I}(x)$ with $\lim_n m(I_n) = 0$, we have

$$\frac{1}{m(I_n)} \int_{I_n} |f(t) - f(x)| dt \leq \frac{1}{m(I_n)} \int_{I_n} |f(t) - g(t)| dt \quad (1)$$

$$+ \frac{1}{m(I_n)} \int_{I_n} |g(t) - g(x)| dt \quad (2)$$

$$+ |g(x) - f(x)| \quad (3)$$

by triangle inequality, adding/subtracting $g(t), g(x)$. We know that when n sufficiently large such that $m(I_n) < \alpha$, $|g(t) - g(x)| \leq \frac{1}{3k} \forall t \in I_n$, hence (2) $\leq \frac{1}{3k}$ for sufficiently large n . For x to be in B_k , we need too that $\limsup_n ((1) + (2) + (3)) > \frac{1}{k}$. But we know that (2) $\leq \frac{1}{3k}$ for all sufficiently large n , we must have that $\limsup_n ((1) + (3)) > \frac{2}{3k}$.

Let

$$C_k := \left\{ x \in \mathbb{R} : \limsup_n (1) > \frac{1}{3k} \right\}, \quad D_k := \left\{ x \in \mathbb{R} : \limsup_n (3) > \frac{1}{3k} \right\},$$

then remark $m(B_k) \leq m(C_k) + m(D_k)$ since $B_k \subseteq C_k \cup D_k$. Then,

$$m(D_k) = m\left(\left\{|f - g| > \frac{1}{3k}\right\}\right) \leq 3k \|f - g\|_1 \leq 3k\varepsilon,$$

by Markov's, and

$$\begin{aligned} m(C_k) &= m\left(\left\{\limsup_n \frac{1}{m(I_n)} \int_{I_n} |f - g| > \frac{1}{3k}\right\}\right) \\ &\leq m\left(\left\{(f - g)^* > \frac{1}{3k}\right\}\right) \leq 3 \cdot 3k \|f - g\|_1 = 9k\varepsilon, \end{aligned}$$

by using the previous H-L inequality. Hence, we find

$$m(B_k) \leq 12k\varepsilon,$$

and, sending $\varepsilon \rightarrow 0$ we find $m(B_k) = 0$, completing the proof. ■