MATH455 - Analysis 4

Functional Analysis - Summary

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1 Linear Operators

Definition 1: For X, Y normed vector spaces, $\mathcal{L}(X,Y) := \left\{T : X \to Y \mid \|T\| := \sup_{x \in X} \frac{\|Tx\|_Y}{\|x\|_X} < \infty\right\}$

Theorem 1: $T : X \to Y$ bounded iff continuous; if X, Y Banach, so is $\mathcal{L}(X,Y)$.

Theorem 2:

(i) Any two nvs of the same finite dimension are isomorphic;
(ii) Any finite dimensional space complete, any finite dimensional subspace is closed;
(iii) $\overline{B(0,1)}$ compact in X iff X finite dimensional.

Theorem 3 (Open Mapping): Let $T : X \to Y$ a bounded linear operator where X, Y Banach. Then, if T surjective, T *open*, that is, $T(\mathcal{U})$ open in Y for any $\mathcal{U}$ open in X.

Remark 1: By scaling & translating, openness of an operator is equivalent to proving $T(B_X(0,1))$ contains $B_Y(0,r)$ for some $r > 0$.

Corollary 1: If $T : X \to Y$ bounded, linear and bijective for X, Y Banach, T^{-1} continuous. In particular, if $\left(X, \|\cdot\|_1\right), \left(X, \|\cdot\|_2\right)$ are two Banach spaces such that $\|x\|_2 \leq C\|x\|_1$, then $\|\cdot\|_1, \|\cdot\|_2$ are equivalent.

Theorem 4 (Closed Graph Theorem): Let $T : X \to Y$ where X, Y Banach. Then T continuous iff T is *closed*, i.e. the graph $G(T) := \{(x, Tx) : x \in X\} \subset X \times Y$ is closed in the product topology.

Remark 2: This theorem crucially uses the fact that the norm

$$\|(x,y)\|_* := \|x\|_X + \|y\|_Y$$

(among others) induces the product topology on $X \times Y$, hence in particular such a norm can be used to make $X \times Y$ a nvs.

Theorem 5 (Uniform Boundedness): Let X Banach and Y an nvs, and let $\mathcal{F} \subset \mathcal{L}(X,Y)$ such that $\forall\, x \in X, \exists\, M_x > 0$ s.t. $\|Tx\|_Y \leq M_x \,\forall\, T \in \mathcal{F}$ (that is, $\mathcal{F}$ pointwise bounded). Then, $\mathcal{F}$ uniformly bounded, i.e. there is some $M > 0$ such that $\|T\|_Y \leq M$ for every $T \in \mathcal{F}$.

Remark 3: This is implied by the consequence of the Baire Category theorem that states that if $\mathcal{F} \subset C(X)$ where X a complete metric space and $\mathcal{F}$ pointwise bounded, then there is a nonempty open set $\mathcal{O} \subset X$ such that $\mathcal{F}$ uniformly bounded on $\mathcal{O}$. In the case of a nvs, by linearity, being uniformly bounded on an open set extends to being uniformly bounded on all of X.

Theorem 6 (Banach-Saks-Steinhaus): Let X Banach and Y an nvs, and $\{T_n\} \subset \mathcal{L}(X, Y)$ such that for every $x \in X$, $\lim_n T_n(x)$ exists in Y. Then

(i) $\{T_n\}$ uniformly bounded in $\mathcal{L}(X, Y)$;
(ii) $T \in \mathcal{L}(X, Y)$ where $T(x) := \lim_n T_n(x)$;
(iii) $\|T\| \leq \liminf_n \|T_n\|$.

Remark 4: (i) follows from uniform boundedness, (ii) from just taking sums limits, (iii) from taking lim(inf)its.

2 Hilbert Spaces; Weak Convergence

Theorem 7 (Cauchy-Schwarz): $|(u, v)| \leq \|u\|\|v\|$.

Theorem 8 (Orthogonality): If $M \subset H$ a closed subspace, for every $x \in H$, there is a unique decomposition

$$x = u + v, \qquad u \in M, v \in M^\perp := \{v \in H \mid (v, y) = 0 \, \forall \, y \in M\},$$

and

$$\|x - u\| = \inf_{y \in M} \|x - y\|, \qquad \|x - v\| = \inf_{y \in M^\perp} \|x - y\|.$$

Theorem 9 (Riesz): For $f \in H^* := \mathcal{L}(H, \mathbb{R})$, there is a unique $y \in H$ such that $f(y) = (y, x), \forall \, x \in H$.

Theorem 10 (Bessel's Inequality): If $\{e_n\} \subset H$ orthonormal, then $\sum_{i=1}^{\infty} |(x, e_i)|^2 \leq \|x\|^2$.

Theorem 11 (Equivalent Notions of Orthonormal Basis): If $\{e_n\} \subset H$ orthonormal, TFAE:

(i) if $(x, e_i) = 0$ for every i, $x = 0$;
(ii) *Parseval's identity* holds, $\|x\|^2 = \sum_{i=1}^{\infty} (x, e_i)^2$, for every $x \in H$;
(iii) $\{e_i\}$ a basis for H, that is $x = \sum_{i=1}^{\infty} (x, e_i) e_i$ for every $x \in H$.

Theorem 12: H is separable (has a countable dense subset) iff H has a countable basis.

Theorem 13 (Properties of the Adjoint): For $T : H \to H$, the *adjoint* $T^* : H \to H$ is defined as the operator with the property $(Tx, y) = (x, T^* y)$ for every $x, y \in H$. Then:

- if $T \in \mathcal{L}(H)$ then $T^* \in \mathcal{L}(H)$ and $\|T^*\| = \|T\|$;
- $(T^*)^* = T$;
- $(T + S)^* = T^* + S^*$;
- $(T \circ S)^* = S^* \circ T^*$;
- if $T \in \mathcal{L}(H)$, then
  - $N(T^*) = R(T)^\perp$, and similarly,
  - $N(T) = R(T^*)^\perp$.

  Note that then $R(T)^\perp$ closed, so one finds $\left(R(T)^\perp\right)^\perp = \overline{R(T)}$.

Definition 2 (Weak Convergence): We say $\{x_n\} \subset X$ *converges weakly* to $x \in X$ and write $x_n \rightharpoonup x$ if for every $T \in X^*$, $Tx_n \to Tx$. By Riesz, this is equivalent to saying $(x_n, y) \to (x, y)$ for every $y \in X$.

We define, then, $\sigma(X, X^*)$ to be the weak topology (on X) generated by the collection of functions X^*; i.e., the coarsest topology for which every functional $T \in X^*$ is continuous.

Theorem 14 (Properties of Weak Convergence):
 (i) If $x_n \rightharpoonup x$, then $\{x_n\}$ bounded in H and $\|x\| \leq \liminf_{n\to\infty} \|x_n\|$.
 (ii) If $y_n \to y$ (strongly) and $x_n \rightharpoonup x$ (weakly) then $(x_n, y_n) \to (x, y)$.

Theorem 15 (Helley's Theorem): Let X a separable normed vector space and $\{f_n\} \subset X^*$ such that there is a $C > 0$ such that $|f_n(x)| \leq C\|x\|$ for every $x \in X$ and $n \geq 1$. Then, there is a subsequence $\{f_{n_k}\}$ and $f \in X^*$ such that $f_{n_k}(x) \to f(x)$ for every $x \in X$.

Remark 5: This is just the Arzelà-Ascoli Lemma; by linearity, the uniform boundedness implies uniform Lipschitz continuity and thus equicontinuity.

Theorem 16 (Weak Compactness): Every bounded sequence in H has a weakly converging subsequence.

Remark 6: This is a consequence of Helley's.

3 L^p Spaces

Theorem 17 (Basic Properties of $L^p(\Omega)$):
 (i) (Holder's Inequality) $\|fg\|_1 \leq \|f\|_p \|g\|_q$ for $f \in L^p(\Omega), g \in L^q(\Omega)$ and $\frac{1}{p} + \frac{1}{q} = 1, 1 \leq p \leq q \leq \infty$;
 (ii) (Riesz-Fischer Theorem) $L^p(\Omega)$ is a Banach space for every $1 \leq p \leq \infty$;
 (iii) $C_c(\mathbb{R}^d)$, simple functions, and step functions are all dense in $L^p(\mathbb{R}^d)$ for every finite p;
 (iv) $L^p(\Omega)$ is separable for every finite p;
 (v) If $\Omega \subset \mathbb{R}^d$ has finite measure, then $L^p(\Omega) \subset L^{p'}(\Omega)$ for every $p \geq p'$;
 (vi) If $f \in L^p(\Omega) \cap L^q(\Omega)$ for $1 \leq p \leq q \leq \infty$, then $f \in L^r(\Omega)$ for every $r \in [p, q]$.

Theorem 18 (Riesz Representation for $L^p(\Omega)$): Let $1 \leq p < \infty$ and q the Holder conjugate of p. Then, if $T \in (L^p(\Omega))^*$, there is a unique $g \in L^q(\Omega)$ such that

$$Tf = \int_\Omega fg, \qquad \forall\, f \in L^p(\Omega),$$

and $\|T\| = \|g\|_q$.

Remark 7: When $p = 2 = q$, then $L^p(\Omega)$ is a Hilbert space so this reduces to the typical Hilbert space theory.

Theorem 19 (Weak Convergence in $L^p(\Omega)$):
- Let $p \in (1, \infty)$ and $\{f_n\} \subset L^p(\Omega)$, then by Riesz, $f_n \rightharpoonup f$ iff $\int_\Omega f_n g \to \int_\Omega fg$ for every $g \in L^q(\Omega)$.
- Suppose f_n are bounded and $f \in L^p(\Omega)$, then $f_n \rightharpoonup f$ if and only if $f_n \to f$ pointwise a.e..
- (Radon-Riesz) For $p \in (1, \infty)$, let $\{f_n\} \subset L^p(\Omega)$ such that $f_n \rightharpoonup f$. Then, $f_n \to f$ strongly if and only if $\|f_n\|_p \to \|f\|$.

Theorem 20 (Weak Compactness in $L^p(\Omega)$): Let $p \in (1, \infty)$. Then, every bounded sequence in $L^p(\Omega)$ has a weakly converging subsequence in $L^p(\Omega)$.

Remark 8: This is essentially the same as the Hilbert space proof.

Theorem 21 (Properties of Convolutions):

(i) $(f * g) * h = f * (g * h)$

(ii) With $\tau_z f(x) := f(x - z)$, $\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g)$

(iii) $\text{supp}(f * g) \subseteq \text{supp}(f) + \text{supp}(g) = \{x + y \mid x \in \text{supp}(f), y \in \text{supp}(g)\}$

Theorem 22 (Young's Inequality): Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ for any $p \in [1, \infty]$, then

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p,$$

so in particular $f * g \in L^p(\Omega)$.

Theorem 23 (Derivatives of Convolutions): Let $f \in L^1(\mathbb{R}^d)$ and $g \in C^1(\mathbb{R}^d)$ with $|\partial_i g| \in L^\infty(\mathbb{R}^d)$ for $i = 1, ..., d$. Then, $f * g \in C^1(\mathbb{R}^d)$, and in particular

$$\partial_i(f * g) = f * (\partial_i g).$$

Remark 9: This holds more generally for many different assumptions on f, g but you basically need to be able to apply dominated convergence theorem to pass the limit involved in taking the derivative under the integral sign.

This extends for $g \in C^k(\mathbb{R}^d)$; in particular, if $g \in C^\infty(\mathbb{R}^d)$, then $f * g \in C^\infty(\mathbb{R}^d)$. It also holds for the gradient, i.e. $\nabla(f * g) = f * (\nabla g)$ (where the convolution is component-wise in the gradient vector).

Theorem 24 (Good Kernels): A *good kernel* is a parametrized family of functions $\{\rho_\varepsilon : \varepsilon \in \mathbb{R}\}$ with the properties

(i) $\int_{\mathbb{R}^d} \rho_\varepsilon(y) \, \mathrm{d}y = 1$,

(ii) $\int_{\mathbb{R}^d} |\rho_\varepsilon(y)| \, \mathrm{d}y \leq M$,

(iii) for every $\delta > 0$, $\int_{|y| > \delta} |\rho_\varepsilon(y)| \, \mathrm{d}y \to 0$ as $\varepsilon \to 0^+$.

The canonical, and in particular both smooth and compactly supported, example is

$$\rho(x) := \begin{cases} C \exp\left(-\frac{1}{1 - |x|^2}\right) & \text{if } |x| \leq 1 \\ 0 & \text{o.w.} \end{cases},$$

where $C = C(d)$ a scaling constant such that ρ integrates to 1. Then $\rho_\varepsilon(x) := \left(\frac{1}{\varepsilon^d}\right)\rho\left(\frac{x}{\varepsilon}\right)$ is a good kernel, supported on $B(0, \varepsilon)$. Then:

(i) if $f \in L^\infty(\mathbb{R}^d)$, $f_\varepsilon := \rho_\varepsilon * f$ and f continuous at x, then $f_\varepsilon(x) \to f(x)$ as $\varepsilon \to 0$;

(ii) if $f \in C(\mathbb{R}^d)$ then $f_\varepsilon \to f$ uniformly on compact sets;

(iii) if $f \in L^p(\mathbb{R}^d)$ with p finite, then $f_\varepsilon \to f$ in $L^p(\mathbb{R}^d)$.

Remark 10: Part 3. follows immediately from 2. by density of $C_c(\mathbb{R}^d)$ in $L^p(\mathbb{R}^d)$.

Corollary 2: $C_c^\infty(\mathbb{R}^d)$ dense in $L^p(\mathbb{R}^d)$ for any finite p.

Theorem 25 (Weierstrass Approximation Theorem): Polynomials are dense in $C([a,b])$, i.e. for any $f \in C([a,b])$ and $\eta > 0$, there is a polynomial $p(x)$ such that $\|p - f\|_{L^\infty([a,b])} < \eta$.

Theorem 26 (Strong Compactness): Let $\{f_n\} \subseteq L^p(\mathbb{R}^d)$ for p finite, such that

- $\{f_n\}$ uniformly bounded in $L^p(\mathbb{R}^d)$, and
- $\lim_{|h| \to 0} \|f_n - \tau_h f_n\|_p = 0$ uniformly in n, i.e. for every $\eta > 0$ there is a $\delta > 0$ such that $|h| < \delta$ implies $\|f_n - \tau_h f_n\|_p < \eta$ for every $n \geq 1$.

Then, for every $\Omega \subset \mathbb{R}^d$ of finite measure, there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ in $L^p(\Omega)$.

Remark 11: This is Arzelà-Ascoli in disguise!

4 Fourier Analysis

Definition 3 (Fourier Series): Let $L^2(\mathbb{T}) = \left\{f : \mathbb{T} \to \mathbb{R} \mid \int_{\mathbb{T}} f^2 < \infty\right\}$ equipped with the inner product $(f, g) = \int_{\mathbb{T}} f\overline{g}$. Then, $e_n(x) := e^{2\pi i n x}$, for $n \in \mathbb{Z}$, is an orthonormal basis for $L^2(\mathbb{T})$. The *Fourier coefficients* of a function f are defined then, for $n \in \mathbb{Z}$,

$$\hat{f}(n) = (f, e_n) = \int_{\mathbb{T}} f(x) e^{-2\pi i n x}\, \mathrm{d}x,$$

and so the *complex Fourier series* is defined

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}.$$

Theorem 27 (Riemann-Lebesgue Lemma): If $f \in L^2(\mathbb{T})$, $\lim_{n \to \infty} \left|\hat{f}(n)\right| = 0$.

Remark 12: By expanding the real and complex parts of the coefficients, this also implies

$$\lim_{n \to \infty} \int_{\mathbb{T}} f(x) \sin(2n\pi x)\, \mathrm{d}x = \lim_{n \to \infty} \int_{\mathbb{T}} f(x) \cos(2n\pi x)\, \mathrm{d}x = 0.$$

Definition 4 (Dirichlet Kernel): The *Dirichlet Kernel* is the sequence of functions defined

$$D_N(x) := \sum_{n=-N}^{N} e^{2\pi i n x} = \frac{\sin\left(2\pi\left(N + \frac{1}{2}\right)x\right)}{\sin\left(2\pi\frac{x}{2}\right)}.$$

Then, the partial sum $S_N f(x) := \sum_{n=-N}^{N} \hat{f}(n) e^{2\pi i n x} = (f * D_N)(x)$.

Theorem 28 (Convergence Results):

(i) If $f \in L^2(\mathbb{T})$ and Lipschitz at x_0, then $S_N f(x_0) \to f(x_0)$
(ii) If $f \in L^2(\mathbb{T}) \cap C^2(\mathbb{T})$, then $S_N f \to f$ uniformly on $\mathbb{T}$.

Definition 5 (Fourier Transform): The *Fourier Transform* of $f : \mathbb{R} \to \mathbb{C}$ is defined

$$\hat{f}(\zeta) := \int_{\mathbb{R}} f(x) e^{-2\pi i \zeta x} \, \mathrm{d}x.$$

The *Inverse Fourier Transform* of $f \in L^1(\mathbb{R})$ is defined

$$\check{f}(x) := \int_{\mathbb{R}} f(\zeta) e^{2\pi i \zeta x} \, \mathrm{d}\zeta = \widehat{f(-\,\cdot)}(x).$$

Theorem 29 (Properties of the Fourier Transform): Let $f, g \in L^1(\mathbb{R})$.

(i) $\hat{f}, \check{f} \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$

(ii) $\widehat{\tau_y f}(\zeta) = e^{-2\pi i \zeta y} \hat{f}(\zeta)$, and $\tau_\eta \hat{f}(\zeta) = \widehat{e^{2\pi i \eta(\cdot)} f(\cdot)}(\zeta)$

(iii) $\widehat{f * g} = \hat{f} \cdot \hat{g}$

(iv) $\int_{\mathbb{R}} f(x) \hat{g}(x) \, \mathrm{d}x = \int_{\mathbb{R}} \hat{f}(x) g(x) \, \mathrm{d}x$

(v) Let $h(x) := e^{\pi a x^2}$ for $a > 0$, then $\hat{f}(\zeta) = \frac{1}{\sqrt{a}} e^{-\pi \frac{\zeta^2}{a}}$

Theorem 30 (Fourier Inversion): If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then f agrees almost everywhere with some $f_0 \in C(\mathbb{R})$ and $\hat{\check{f}} = \check{\hat{f}} = f_0$.

Theorem 31 (Plancherel's Theorem): If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$ and $\|f\|_2 = \left\|\hat{f}\right\|_2$.

Remark 13: Using this, one extends the Fourier Transform to $f \in L^2(\mathbb{R})$ by taking a sequence of smooth, compactly supporting functions approximating f in L^2, and taking the limit of the Fourier transforms in $L^2(\mathbb{R})$.

Theorem 32: If $f \in L^1(\mathbb{R})$, $\hat{f} \in C_0(\mathbb{R})$, the space of continuous functions with $|f(x)| \to 0$ as $|x| \to \infty$.

Theorem 33 (Poisson Summation Formula): Let $f \in C(\mathbb{R})$ be such that $|f(x)| \leq C(1 + |x|)^{-(1+\varepsilon)}$ and $\left|\hat{f}(\zeta)\right| \leq C(1 + |\zeta|)^{-(1+\varepsilon)}$ for some constants $C, \varepsilon > 0$. Then, for every $x \in \mathbb{R}$,

$$\sum_{k \in \mathbb{Z}} f(x + k) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k x}.$$

Remark 14: In words, this means the *periodization* (the LHS) of f equals the Fourier series of f.