

MATH455 - Analysis 4

Abstract Metric, Topological Spaces; Functional Analysis.

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§1 ABSTRACT METRIC AND TOPOLOGICAL SPACES

§1.1 Review of Metric Spaces

Throughout fix X a nonempty set.

↪ **Definition 1.1** (Metric): $\rho : X \times X \rightarrow \mathbb{R}$ is called a *metric*, and thus (X, ρ) a *metric space*, if for all $x, y, z \in X$,

- $\rho(x, y) \geq 0$,
- $\rho(x, y) = 0 \Leftrightarrow x = y$,
- $\rho(x, y) = \rho(y, x)$, and
- $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

↪ **Definition 1.2** (Norm): Let X a linear space. A function $\| \cdot \| : X \rightarrow [0, \infty)$ is called a *norm* if for all $u, v \in X$ and $\alpha \in \mathbb{R}$,

- $\|u\| = 0 \Leftrightarrow u = 0$,
- $\|u + v\| \leq \|u\| + \|v\|$, and
- $\|\alpha u\| = |\alpha| \|u\|$.

Remark 1.1: A norm induces a metric by $\rho(x, y) := \|x - y\|$.

↪ **Definition 1.3:** Given two metrics ρ, σ on X , we say they are *equivalent* if $\exists C > 0$ such that $\frac{1}{C}\sigma(x, y) \leq \rho(x, y) \leq C\sigma(x, y)$ for every $x, y \in X$. A similar definition follows for equivalence of norms.

Given a metric space (X, ρ) , then, we have the notion of

- open balls $B(x, r) = \{y \in X : \rho(x, y) < r\}$,
- open sets (subsets of X with the property that for every $x \in X$, there is a constant $r > 0$ such that $B(x, r) \subseteq X$), closed sets, closures, and
- *convergence*.

↪ **Definition 1.4** (Convergence): $\{x_n\} \subseteq X$ converges to $x \in X$ if $\lim_{n \rightarrow \infty} \rho(x_n, x) = 0$.

We have several (equivalent) notions, then, of continuity; via sequences, $\varepsilon - \delta$ definition, and by pullbacks (inverse images of open sets are open).

↪ **Definition 1.5** (Uniform Continuity): $f : (X, \rho) \rightarrow (Y, \sigma)$ uniformly continuous if f has a “modulus of continuity”, i.e. there is a continuous function $\omega : [0, \infty) \rightarrow [0, \infty)$ such that

$$\sigma(f(x_1), f(x_2)) \leq \omega(\rho(x_1, x_2))$$

for every $x_1, x_2 \in X$.

Remark 1.2: For instance, we say f Lipschitz continuous if there is a constant $C > 0$ such that $\omega(\cdot) = C(\cdot)$. Let $\alpha \in (0, 1)$. We say f α -Holder continuous if $\omega(\cdot) = C(\cdot)^\alpha$ for some constant C .

↪ **Definition 1.6** (Completeness): We say (X, ρ) *complete* if every Cauchy sequence in (X, ρ) converges to a point in X .

Remark 1.3: If (X, ρ) complete and $E \subseteq X$, then (E, ρ) is complete iff E closed in X .

§1.2 Compactness, Separability

↪ **Definition 1.7** (Open Cover, Compactness): $\{X_\lambda\}_{\lambda \in \Lambda} \subseteq 2^X$, where X_λ open in X and Λ an arbitrary index set, an *open cover* of X if for every $x \in X$, $\exists \lambda \in \Lambda$ such that $x \in X_\lambda$.

X is *compact* if every open cover of X admits a compact subcover. We say $E \subseteq X$ compact if (E, ρ) compact.

↪ **Definition 1.8** (Totally Bounded, ε -nets): (X, ρ) *totally bounded* if $\forall \varepsilon > 0$, there is a finite cover of X of balls of radius ε . If $E \subseteq X$, an ε -*net* of E is a collection $\{B(x_i, \varepsilon)\}_{i=1}^N$ such that $E \subseteq \bigcup_{i=1}^N B(x_i, \varepsilon)$ and $x_i \in X$ (note that x_i need not be in E).

↪ **Definition 1.9** (Sequentially Compact): (X, ρ) *sequentially compact* if every sequence in X has a convergent subsequence whose limit is in X .

↪ **Definition 1.10** (Relatively / Pre- Compact): $E \subseteq X$ *relatively compact* if \overline{E} compact.

↪ **Theorem 1.1:** TFAE:

1. X complete and totally bounded;
2. X compact;
3. X sequentially compact.

Remark 1.4: $E \subseteq X$ relatively compact if every sequence in E has a convergent subsequence.

Let $f : (X, \rho) \rightarrow (Y, \sigma)$ continuous with (X, ρ) compact. Then,

- $f(X)$ compact in Y ;
- if $Y = \mathbb{R}$, the max and min of f over X are achieved;
- f is uniformly continuous.

Let $C(X) := \{f : X \rightarrow \mathbb{R} \mid f \text{ continuous}\}$ and $\|f\|_\infty := \max_{x \in X} |f(x)|$ the sup (max, in this case) norm. Then,

Theorem 1.2: Let (X, ρ) compact. Then, $(C(X), \|\cdot\|_\infty)$ is complete.

PROOF. Let $\{f_n\} \subseteq C(X)$ Cauchy with respect to $\|\cdot\|_\infty$. Then, there exists a subsequence $\{f_{n_k}\}$ such that for each $k \geq 1$, $\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq 2^{-k}$ (to construct this subsequence, let $n_1 \geq 1$ be such that $\|f_n - f_{n_1}\|_\infty < \frac{1}{2}$ for all $n \geq n_1$, which exists since $\{f_n\}$ Cauchy. Then, for each $k \geq 1$, define inductively n_{k+1} such that $n_{k+1} > n_k$ and $\|f_n - f_{n_{k+1}}\|_\infty < \frac{1}{2^{k+1}}$ for each $n \geq n_{k+1}$. Then, for any $k \geq 1$, $\|f_{n_{k+1}} - f_{n_k}\|_\infty < 2^{-k}$, since $n_{k+1} > n_k$).

Let $j \in \mathbb{N}$. Then, for any $k \geq 1$,

$$\|f_{n_{k+j}} - f_{n_k}\|_\infty \leq \sum_{\ell=k}^{k+j-1} \|f_{n_{\ell+1}} - f_{n_\ell}\|_\infty \leq \sum_{\ell=k}^{\infty} 2^{-\ell}$$

and hence for each $x \in X$, with $c_k := f_{n_k}(x)$,

$$|c_{k+j} - c_k| \leq \sum_{\ell=k}^{\infty} 2^{-\ell}.$$

The RHS is the tail of a converging series, and thus $|c_{k+j} - c_k| \rightarrow 0$ as $k \rightarrow \infty$ i.e. $\{c_k\}$ a Cauchy sequence, in \mathbb{R} . $(\mathbb{R}, |\cdot|)$ complete, so $\lim_{k \rightarrow \infty} c_k =: f(x)$ exists for each $x \in X$. So, for each $x \in X$, we find

$$|f_{n_k}(x) - f(x)| \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

and since the RHS is independent of x , we may pass to the sup norm, and find

$$\|f_{n_k} - f\|_\infty \leq \sum_{\ell=k}^{\infty} 2^{-\ell},$$

with the RHS $\rightarrow 0$ as $k \rightarrow \infty$. Hence, $f_{n_k} \rightarrow f$ in $C(X)$ as $k \rightarrow \infty$. In other words, we have uniform convergence of $\{f_{n_k}\}$. Each $\{f_{n_k}\}$ continuous, and thus f also continuous, and thus $f \in C(X)$.

It remains to show convergence along the whole sequence. Suppose otherwise. Then, there is some $\alpha > 0$ and a subsequence $\{f_{n_j}\} \subseteq \{f_n\}$ such that $\|f_{n_j} - f\|_\infty > \alpha$.

$\alpha > 0$ for every $j \geq 1$. Then, let k be sufficiently large such that $\|f - f_{n_k}\|_\infty \leq \frac{\alpha}{2}$. Then, for every $j \geq 1$ and k sufficiently large,

$$\begin{aligned}\|f_{n_j} - f_{n_k}\|_\infty &\geq \|f_{n_j} - f\|_\infty - \|f - f_{n_k}\|_\infty \\ &> \alpha - \frac{\alpha}{2} > 0,\end{aligned}$$

which contradicts the Cauchy-ness of $\{f_n\}$, completing the proof. ■

↪ **Definition 1.11** (Density/Separability): A set $D \subseteq X$ is called *dense* in X if for every nonempty open subset $A \subseteq X$, $D \cap A \neq \emptyset$. We say X *separable* if there is a countable dense subset of X .

Remark 1.5: If A dense in X , then $\overline{A} = X$.

↪ **Proposition 1.1:** If X compact, X separable.

PROOF. Since X compact, it is totally bounded. So, for $n \in \mathbb{N}$, there is some K_n and $\{x_i\} \subseteq X$ such that $X \subseteq \bigcup_{i=1}^{K_n} B(x_i, \frac{1}{n})$. Then, $D = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{K_n} \{x_i\}$ countable and dense in X . ■

§1.3 Arzelà-Ascoli

The goal in this section is to find conditions for a sequence of functions $\{f_n\} \subseteq C(X)$ to be precompact, namely, to have a uniformly convergent subsequence.

↪ **Corollary 1.1:** Any Cauchy sequence converges if it has a convergent subsequence.

PROOF. Let $\{x_n\}$ be a Cauchy sequence in a metric space (X, ρ) with convergent subsequence $\{x_{n_k}\}$ which converges to some $x \in X$. Fix $\varepsilon > 0$. Let $N \geq 1$ be such that if $m, n \geq N$, $\rho(x_n, x_m) < \frac{\varepsilon}{2}$. Let $K \geq 1$ be such that if $k \geq K$, $\rho(x_{n_k}, x) < \frac{\varepsilon}{2}$. Let $n, n_k \geq \max\{N, K\}$, then

$$\rho(x, x_n) \leq \rho(x, x_{n_k}) + \rho(x_{n_k}, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
■

↪ **Definition 1.12** (Equicontinuous): A family $\mathcal{F} \subseteq C(X)$ is called *equicontinuous* at $x \in X$ if $\forall \varepsilon > 0$ there exists a $\delta = \delta(x, \varepsilon) > 0$ such that if $\rho(x, x') < \delta$ then $|f(x) - f(x')| < \varepsilon$ for every $f \in \mathcal{F}$.

Remark 1.6: \mathcal{F} equicontinuous at x iff every $f \in \mathcal{F}$ share the same modulus of continuity.

↪ **Definition 1.13** (Pointwise/uniformly bounded): $\{f_n\}$ pointwise bounded if $\forall x \in X$, $\exists M(x) > 0$ such that $|f_n(x)| \leq M(x) \forall n$, and uniformly bounded if such an M exists independent of x .

↪ **Lemma 1.1** (Arzelà-Ascoli Lemma): Let X separable and let $\{f_n\} \subseteq C(X)$ be pointwise bounded and equicontinuous. Then, there is a function f and a subsequence $\{f_{n_k}\}$ which converges pointwise to f on all of X .

PROOF. Let $D = \{x_j\}_{j=1}^{\infty} \subseteq X$ be a countable dense subset of X . Since $\{f_n\}$ p.w. bounded, $\{f_n(x_1)\}$ as a sequence of real numbers is bounded and so by the Bolzano-Weierstrass (BW) Theorem there is a convergent subsequence $\{f_{n(1,k)}(x_1)\}_k$ that converges to some $a_1 \in \mathbb{R}$. Consider now $\{f_{n(1,k)}(x_2)\}_k$, which is again a bounded sequence of \mathbb{R} and so has a convergent subsequence, call it $\{f_{n(2,k)}(x_2)\}_k$ which converges to some $a_2 \in \mathbb{R}$. Note that $\{f_{n(2,k)}\} \subseteq \{f_{n(1,k)}\}$, so also $f_{n(2,k)}(x_1) \rightarrow a_1$ as $k \rightarrow \infty$. We can repeat this procedure, producing a sequence of real numbers $\{a_\ell\}$, and for each $j \in \mathbb{N}$ a subsequence $\{f_{n(j,k)}\}_k \subseteq \{f_n\}$ such that $f_{n(j,k)}(x_\ell) \rightarrow a_\ell$ for each $1 \leq \ell \leq j$. Define then

$$f : D \rightarrow \mathbb{R}, f(x_j) := a_j.$$

Consider now

$$f_{n_k} := f_{n(k,k)}, k \geq 1,$$

the “diagonal sequence”, and remark that $f_{n_k}(x_j) \rightarrow a_j = f(x_j)$ as $k \rightarrow \infty$ for every $j \geq 1$. Hence, $\{f_{n_k}\}_k$ converges to f on D , pointwise.

We claim now that $\{f_{n_k}\}$ converges on all of X to some function $f : X \rightarrow \mathbb{R}$, pointwise. Put $g_k := f_{n_k}$ for notational convenience. Fix $x_0 \in X$, $\varepsilon > 0$, and let $\delta > 0$ be such that if $x \in X$ such that $\rho(x, x_0) < \delta$, $|g_k(x) - g_k(x_0)| < \frac{\varepsilon}{3}$ for every $k \geq 1$, which exists by equicontinuity. Since D dense in X , there is some $x_j \in D$ such that $\rho(x_j, x_0) < \delta$. Then, since $g_k(x_j) \rightarrow f(x_j)$ (pointwise), $\{g_k(x_j)\}_k$ is Cauchy and so there is some $K \geq 1$ such that for every $k, \ell \geq K$, $|g_\ell(x_j) - g_k(x_j)| < \frac{\varepsilon}{3}$. And hence, for every $k, \ell \geq K$,

$$|g_k(x_0) - g_\ell(x_0)| \leq |g_k(x_0) - g_k(x_j)| + |g_k(x_j) - g_\ell(x_j)| + |g_\ell(x_j) - g_\ell(x_0)| < \varepsilon,$$

so namely $\{g_k(x_0)\}_k$ Cauchy as a sequence in \mathbb{R} . Since \mathbb{R} complete, then $\{g_k(x_0)\}_k$ also converges, to, say, $f(x_0) \in \mathbb{R}$. Since x_0 was arbitrary, this means there is some function $f : X \rightarrow \mathbb{R}$ such that $g_k \rightarrow f$ pointwise on X as we aimed to show. ■

↪ **Definition 1.14** (Uniformly Equicontinuous): $\mathcal{F} \subseteq C(X)$ is said to be uniformly equicontinuous if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\forall x, y \in X$ with $\rho(x, y) < \delta$, $|f(x) - f(y)| < \varepsilon$ for every $f \in \mathcal{F}$. That is, every function in \mathcal{F} has the same modulus of continuity.

↪ **Proposition 1.2** (Sufficient Conditions for Uniform Equicontinuity):

1. $\mathcal{F} \subseteq C(X)$ uniformly Lipschitz
2. $\mathcal{F} \subseteq C(X) \cap C^1(X)$ has a uniform L^∞ bound on the first derivative
3. $\mathcal{F} \subseteq C(X)$ uniformly Holder continuous
4. (X, ρ) compact and \mathcal{F} equicontinuous

PROOF.

1. If $C > 0$ is such that $|f(x) - f(y)| \leq C\rho(x, y)$ for every $x, y \in X$ and $f \in \mathcal{F}$, then for $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{C}$, then if $\rho(x, y) \leq \delta$, $|f(x) - f(y)| \leq C\delta < \varepsilon$, and δ independent of x (and f) since it only depends on C which is independent of x, y, f , etc.
3. Akin to 1.

■

↪ **Theorem 1.3** (Arzelà-Ascoli): Let (X, ρ) a compact metric space and $\{f_n\} \subseteq C(X)$ be a uniformly bounded and (uniformly) equicontinuous family of functions. Then, $\{f_n\}$ is pre-compact in $C(X)$, i.e. there exists $\{f_{n_k}\} \subseteq \{f_n\}$ such that f_{n_k} is uniformly convergent on X .

PROOF. Since (X, ρ) compact it is separable and so by the lemma there is a subsequence $\{f_{n_k}\}$ that converges pointwise on X . Denote by $g_k := f_{n_k}$ for notational convenience.

We claim $\{g_k\}$ uniformly Cauchy. Let $\varepsilon > 0$. By uniform equicontinuity, there is a $\delta > 0$ such that $\rho(x, y) < \delta \Rightarrow |g_k(x) - g_k(y)| < \frac{\varepsilon}{3}$. Since X compact it is totally bounded so there exists $\{x_i\}_{i=1}^N$ such that $X \subseteq \bigcup_{i=1}^N B(x_i, \delta)$. For every $1 \leq i \leq N$, $\{g_k(x_i)\}$ converges by the lemma hence is Cauchy in \mathbb{R} . So, there exists a K_i such that for every $k, \ell \geq K_i$ $|g_k(x_i) - g_\ell(x_i)| \leq \frac{\varepsilon}{3}$. Let $K := \max\{K_i\}$. Then for every $\ell, k \leq K$, $|g_k(x_i) - g_\ell(x_i)| \leq \frac{\varepsilon}{3}$ for every $i = 1, \dots, N$. So, for all $x \in X$, there is some x_i such that $\rho(x, x_i) < \delta$, and so for every $k, \ell \geq K$,

$$\begin{aligned} |g_k(x) - g_\ell(x)| &\leq |g_k(x) - g_k(x_i)| \\ &\quad + |g_k(x_i) - g_\ell(x_i)| \\ &\quad + |g_\ell(x_i) - g_\ell(x)| < \varepsilon, \end{aligned}$$

the first and last follow by the equicontinuity and the second from the lemma. This holds for every x and thus $\|g_k - g_\ell\|_\infty < \varepsilon$, so $\{g_k\}$ Cauchy in $C(X)$. But $C(X)$ complete so converges in the space.

■

Remark 1.7: If $K \subseteq X$ a compact set, then K bounded and closed.

↪ **Theorem 1.4:** Let (X, ρ) compact and $\mathcal{F} \subseteq C(X)$. Then, \mathcal{F} a compact subspace of $C(X)$ iff \mathcal{F} closed, uniformly bounded, and (uniformly) equicontinuous.

PROOF. (\Leftarrow) Let $\{f_n\} \subseteq \mathcal{F}$. By Arzelà-Ascoli Theorem, there exists a subsequence $\{f_{n_k}\}$ that converges uniformly to some $f \in C(X)$. Since \mathcal{F} closed, $f \in \mathcal{F}$ and so \mathcal{F} sequentially compact hence compact.

(\Rightarrow) \mathcal{F} compact so closed and bounded in $C(X)$. To prove equicontinuous, we argue by contradiction. Suppose otherwise, that \mathcal{F} not-equicontinuous at some $x \in X$. Then, there is some $\varepsilon_0 > 0$ and $\{f_n\} \subseteq \mathcal{F}$ and $\{x_n\} \subseteq X$ such that $|f_n(x_n) - f_n(x)| \geq \varepsilon_0$ while $\rho(x, x_n) < \frac{1}{n}$. Since $\{f_n\}$ bounded and \mathcal{F} compact, there is a subsequence $\{f_{n_k}\}$ that converges to f uniformly. Let K be such that $\forall k \geq K, \|f_{n_k} - f\|_\infty \leq \frac{\varepsilon_0}{3}$. Then,

$$\begin{aligned} |f(x_{n_k}) - f| &\geq |f(x_{n_k}) - f_{n_k}(x_{n_k})| - |f_{n_k}(x_{n_k}) - f_{n_k}(x)| - |f_{n_k}(x) - f(x)| \\ &\geq \frac{\varepsilon_0}{3}, \end{aligned}$$

while $\rho(x_{n_k}, x) \leq \frac{1}{n_k}$, so f cannot be continuous at x , a contradiction. ■

§1.4 Baire Category Theorem

↪ **Definition 1.15** (Hollow/Nowhere Dense): We say a set $E \subseteq X$ *hollow* if $\text{int}(E) = \emptyset$. We say a set $E \subseteq X$ *nowhere dense* if its closure is hollow, i.e. $\text{int}(\overline{E}) = \emptyset$.

Remark 1.8: Notice that E hollow $\Leftrightarrow E^c$ dense, since $\text{int}(E) = \emptyset \Rightarrow (\text{int}(E))^c = \overline{E^c} = X$.

↪ **Theorem 1.5** (Baire Category Theorem): Let X be a complete metric space.

- (a) Let $\{F_n\}$ a collection of closed hollow sets. Then, $\bigcup_{n=1}^{\infty} F_n$ also hollow.
- (b) Let $\{\mathcal{O}_n\}$ a collection of open dense sets. Then, $\bigcap_{n=1}^{\infty} \mathcal{O}_n$ also dense.

PROOF. Notice that (a) \Leftrightarrow (b) by taking complements. We prove (b).

Put $G := \bigcap_{n=1}^{\infty} \mathcal{O}_n$. Fix $x \in X$ and $r > 0$, then to show density of G is to show $G \cap B(x, r) \neq \emptyset$.

Since \mathcal{O}_1 dense, then $\mathcal{O}_1 \cap B(x, r)$ nonempty and in particular open. So, let $x_1 \in X$ and $r_1 < \frac{1}{2}$ such that $\overline{B}(x_1, r_1) \subseteq B(x, 2r_1) \subseteq \mathcal{O}_1 \cap B(x, r)$.

Similarly, since \mathcal{O}_2 dense, $\mathcal{O}_2 \cap B(x_1, r_1)$ open and nonempty so there exists $x_2 \in X$ and $r_2 < 2^{-2}$ such that $\overline{B}(x_2, r_2) \subseteq \mathcal{O}_2 \cap B(x_1, r_1)$.

Repeat in this manner to find $x_n \in X$ with $r_n < 2^{-n}$ such that $\overline{B}(x_n, r_n) \subseteq \mathcal{O}_n \cap B(x_{n-1}, r_{n-1})$ for any $n \in \mathbb{N}$. This creates a sequence of sets

$$\overline{B}(x_1, r_1) \supseteq \overline{B}(x_2, r_2) \supseteq \cdots,$$

with $r_n \rightarrow 0$. Hence, the sequence of points $\{x_n\}$ is Cauchy and since X is complete, $x_j \rightarrow x_0 \in X$, so in particular

$$\{x_0\} = \bigcap_{n=1}^{\infty} \overline{B}(x_n, r_n),$$

hence $x_0 \in \mathcal{O}_n$ for every n and thus $G \cap B(x, r)$ is nonempty. ■

Corollary 1.2: Let X be complete and $\{F_n\}$ a sequence of closed sets in X . If $X = \bigcup_{n \geq 1} F_n$, there is some n_0 such that $\text{int}(F_{n_0}) \neq \emptyset$.

PROOF. If not, it violates BCT since X is not hollow in itself; $\text{int}(X) = X$. ■

Corollary 1.3: Let X be complete and $\{F_n\}$ a sequence of closed sets in X . Then, $\bigcup_{n=1}^{\infty} \partial F_n$ is hollow.

PROOF. We claim $\text{int}(\partial F_n) = \emptyset$. Suppose not, then there exists some $B(x_0, r) \subseteq \partial F_n$. Then $x_0 \in \partial F_n$ but $B(x_0, r) \cap F_n^c = \emptyset$, a contradiction. So, since ∂F_n is closed and $\partial F_n \cap B(x_0, r) = \emptyset$ for every such ball, by BCT $\bigcup_{n=1}^{\infty} \partial F_n$ must be hollow. ■

1.4.1 Applications of Baire Category Theorem

Theorem 1.6: Let $\mathcal{F} \subset C(X)$ where X is complete. Suppose \mathcal{F} is pointwise bounded. Then, there exists a nonempty, open set $\mathcal{O} \subseteq X$ such that \mathcal{F} is uniformly bounded on \mathcal{O} .

PROOF. Let

$$\begin{aligned} E_n &:= \{x \in X : |f(x)| \leq n \forall f \in \mathcal{F}\} \\ &= \bigcap_{f \in \mathcal{F}} \underbrace{\{x : |f(x)| \leq n\}}_{\text{closed}}. \end{aligned}$$

Since \mathcal{F} is pointwise bounded, for every $x \in X$ there is some $M_x > 0$ such that $|f(x)| \leq M_x$ for every $f \in \mathcal{F}$. Hence, for every $n \in \mathbb{N}$ such that $n \geq M_x$, $x \in E_n$ and thus $X = \bigcup_{n=1}^{\infty} E_n$.

E_n is closed and hence by the previous corollaries there is some n_0 such that $\text{int}(E_{n_0}) \neq \emptyset$ and hence there is some $r > 0$ and $x_0 \in X$ such that $B(x_0, r) \subseteq E_{n_0}$. Then, for every $x \in B(x_0, r)$, $|f(x)| \leq n_0$ for every $f \in \mathcal{F}$, which gives our desired non-empty open set upon which \mathcal{F} is uniformly bounded. ■

↪ **Theorem 1.7:** Let X complete, and $\{f_n\} \subseteq C(X)$ such that $f_n \rightarrow f$ pointwise on X . Then, there exists a dense subset $D \subseteq X$ such that $\{f_n\}$ equicontinuous on D and f continuous on D .

PROOF. For $m, n \in \mathbb{N}$, let

$$\begin{aligned} E(m, n) &:= \left\{ x \in X : |f_j(x) - f_k(x)| \leq \frac{1}{m} \forall j, k \geq n \right\} \\ &= \bigcap_{j, k \geq n} \left\{ x : |f_j(x) - f_k(x)| \leq \frac{1}{m} \right\}. \end{aligned}$$

The union of the boundaries of these sets are hollow, hence $D := \left(\bigcup_{m, n \geq 1} \partial E(m, n) \right)^c$ is dense. Then, if $x \in D \cap E(m, n)$, then $x \in (\partial E(m, n))^c$ implies $x \in \text{int}(E(m, n))$.

We claim $\{f_n\}$ equicontinuous on D . Let $x_0 \in D$ and $\varepsilon > 0$. Let $\frac{1}{m} \leq \frac{\varepsilon}{4}$. Then, since $\{f_n(x_0)\}$ convergent it is therefore Cauchy (in \mathbb{R}). Hence, there is some N such that $|f_j(x_0) - f_k(x_0)| \leq \frac{1}{m}$ for every $j, k \geq N$, so $x_0 \in D \cap E(m, N)$ hence $x_0 \in \text{int}(E(m, N))$.

Let $B(x_0, r) \subseteq E(m, N)$. Since f_N continuous at x_0 there is some $\delta > 0$ such that $\delta < r$ and

$$|f_N(x) - f_N(x_0)| < \frac{1}{m} \forall x \in B(x_0, \delta),$$

and hence

$$\begin{aligned} |f_j(x) - f_j(x_0)| &\leq |f_j(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f_j(x_0)| \\ &\leq \frac{3}{m} \leq \frac{3}{4}\varepsilon, \end{aligned}$$

for every $x \in B(x_0, \delta)$ and $j \geq N$, where the first, last bounds come from Cauchy and the middle from continuity of f_N . Hence, we've show $\{f_n\}$ equicontinuous at x_0 since δ was independent of f .

In particular, this also gives for every $x \in B(x_0, \delta)$ the limit

$$\frac{3}{4}\varepsilon > \lim_{j \rightarrow \infty} |f_j(x) - f_j(x_0)| = |f(x) - f(x_0)|,$$

so f continuous on D . ■

§1.5 Topological Spaces

Throughout, assume $X \neq \emptyset$.

↪ **Definition 1.16** (Topology): Let $X \neq \emptyset$. A *topology* \mathcal{T} on X is a collection of subsets of X , called *open sets*, such that

- $X, \emptyset \in \mathcal{T}$;
- If $\{E_n\} \subseteq \mathcal{T}$, $\bigcap_{n=1}^N E_n \in \mathcal{T}$ (closed under *finite* intersections);
- If $\{E_n\} \subseteq \mathcal{T}$, $\bigcup_n E_n \in \mathcal{T}$ (closed under *arbitrary* unions).

If $x \in X$, a set $E \in \mathcal{T}$ containing x is called a neighborhood of x .

↪ **Proposition 1.3**: $E \subseteq X$ open \Leftrightarrow for every $x \in E$, there is a neighborhood of x contained in E .

PROOF. \Rightarrow is trivial by taking the neighborhood to be E itself. \Leftarrow follows from the fact that, if for each x we let \mathcal{U}_x a neighborhood of x contained in E , then

$$E = \bigcup_{x \in E} \mathcal{U}_x,$$

so E open being a union of open sets. ■

⊗ **Example 1.1**: Every metric space induces a natural topology given by open sets under the metric. The *discrete topology* is given by $\mathcal{T} = 2^X$ (and is actually induced by the discrete metric), and is the largest topology. The *trivial topology* $\{\emptyset, X\}$ is the smallest. The *relative topology* defined on a subset $Y \subseteq X$ is given by $\mathcal{T}_Y := \{E \cap Y : E \in \mathcal{T}\}$.

↪ **Definition 1.17** (Base): Given a topological space (X, \mathcal{T}) , let $x \in X$. A collection \mathcal{B}_x of neighborhoods of x is called a *base* of \mathcal{T} at x if for every neighborhood \mathcal{U} of x , there is a set $B \in \mathcal{B}_x$ such that $B \subseteq \mathcal{U}$.

We say a collection \mathcal{B} a base for all of \mathcal{T} if for every $x \in X$, there is a base for x , $\mathcal{B}_x \subseteq \mathcal{B}$.

↪ **Proposition 1.4**: If (X, \mathcal{T}) a topological space, then $\mathcal{B} \subseteq \mathcal{T}$ a base for \mathcal{T} \Leftrightarrow every nonempty open set $\mathcal{U} \in \mathcal{T}$ can be written as a union of elements of \mathcal{B} .

PROOF. \Rightarrow If \mathcal{U} open, then for $x \in \mathcal{U}$ there is some basis element B_x contained in \mathcal{U} . So in particular $\mathcal{U} = \bigcup_{x \in \mathcal{U}} B_x$.

\Leftarrow Let $x \in \mathcal{U}$ and $\mathcal{B}_x := \{B \in \mathcal{B} \mid x \in B\}$. Then, for every neighborhood of x , there is some B in \mathcal{B}_x such that $B \subseteq \mathcal{U}$ so \mathcal{B}_x a base for \mathcal{T} at x . ■

Remark 1.9: A base \mathcal{B} defines a unique topology, $\{\emptyset, \cup \mathcal{B}_x\}$.

↪ **Proposition 1.5:** $\mathcal{B} \subseteq 2^X$ a base for a topology on $X \Leftrightarrow$

- $X = \bigcup_{B \in \mathcal{B}} B$
- If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a $B \in \mathcal{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

PROOF. (\Rightarrow) If \mathcal{B} a base, then X open so $X = \bigcup_B B$. If $B_1, B_2 \in \mathcal{B}$, then $B_1 \cap B_2$ open so there must exist some $B \subseteq B_1 \cap B_2$ in \mathcal{B} .

(\Leftarrow) Let

$$\mathcal{T} = \{\mathcal{U} \mid \forall x \in \mathcal{U}, \exists B \in \mathcal{B} \text{ with } x \in B \subseteq \mathcal{U}\}.$$

One can show this a topology on X with \mathcal{B} as a base. ■

↪ **Definition 1.18:** If $\mathcal{T}_1 \subsetneq \mathcal{T}_2$, we say \mathcal{T}_1 *weaker/coarser* and \mathcal{T}_2 *stronger/finer*.

Given a subset $S \subseteq 2^X$, define

$$\mathcal{T}(S) = \bigcap \text{all topologies containing } S = \text{unique weakest topology containing } S$$

to be the topology *generated* by S .

↪ **Proposition 1.6:** If $S \subseteq 2^X$,

$$\mathcal{T}(S) = \bigcup \{\text{finite intersections of elts of } S\}.$$

We call S a “subbase” for $\mathcal{T}(S)$ (namely, we allow finite intersections of elements in S to serve as a base for $\mathcal{T}(S)$).

PROOF. Let $\mathcal{B} := \{X, \text{finite intersections of elements of } S\}$. We claim this a base for $\mathcal{T}(S)$. ■

↪ **Definition 1.19** (Point of closure/accumulation point): If $E \subseteq X, x \in X$, x is called a *point of closure* if $\forall \mathcal{U}_x, \mathcal{U}_x \cap E \neq \emptyset$. The collection of all such sets is called the *closure* of E , denoted \overline{E} . We say E *closed* if $E = \overline{E}$.

↪ **Proposition 1.7:** Let $E \subseteq X$, then

- \overline{E} closed,
- \overline{E} is the smallest closed set containing E ,
- E open $\Leftrightarrow E^c$ closed.

§1.6 Separation, Countability, Separability

↪ **Definition 1.20:** A neighborhood of a set $K \subseteq X$ is any open set containing K .

↪ **Definition 1.21** (Notions of Separation): We say (X, \mathcal{T}) :

- *Tychonoff Separable* if $\forall x, y \in X, \exists \mathcal{U}_x, \mathcal{U}_y$ such that $y \notin \mathcal{U}_x, x \notin \mathcal{U}_y$
- *Hausdorff Separable* if $\forall x, y \in X$ can be separated by two disjoint open sets i.e. $\exists \mathcal{U}_x \cap \mathcal{U}_y = \emptyset$
- *Normal* if Tychonoff and in addition any 2 disjoint closed sets can be separated by disjoint neighborhoods.

Remark 1.10: Metric space \subseteq normal space \subseteq Hausdorff space \subseteq Tychonoff space.

↪ **Proposition 1.8:** Tychonoff $\Leftrightarrow \forall x \in X, \{x\}$ closed.

PROOF. For every $x \in X$,

$$\begin{aligned} \{x\} \text{ closed} &\Leftrightarrow \{x\}^c \text{ open} \\ &\Leftrightarrow \forall y \in \{x\}^c, \exists \mathcal{U}_y \subseteq \{x\}^c \\ &\Leftrightarrow \forall y \neq x, \exists \mathcal{U}_y \text{ s.t. } x \notin \mathcal{U}_y, \end{aligned}$$

and since this holds for every x , X Tychonoff. ■

↪ **Proposition 1.9:** Every metric space normal.

PROOF. Define, for $F \subseteq X$, the function

$$\text{dist}(F, x) := \inf\{\rho(x, x') \mid x' \in F\}.$$

Notice that if F closed and $x \notin F$, then $\text{dist}(F, x) > 0$ (since F^c open so there exists some $B(x, \varepsilon) \subseteq F^c$ so $\rho(x, x') \geq \varepsilon$ for every $x' \in F$). Let F_1, F_2 be closed disjoint sets, and define

$$\begin{aligned} \mathcal{O}_1 &:= \{x \in X \mid \text{dist}(F_1, x) < \text{dist}(F_2, x)\}, \\ \mathcal{O}_2 &:= \{x \in X \mid \text{dist}(F_1, x) > \text{dist}(F_2, x)\}. \end{aligned}$$

Then, $F_1 \subseteq \mathcal{O}_1, F_2 \subseteq \mathcal{O}_2$, and $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. If we show $\mathcal{O}_1, \mathcal{O}_2$ open, we'll be done.

Let $x \in \mathcal{O}_1$ and $\varepsilon > 0$ such that $\text{dist}(F_1, x) + \varepsilon \leq \text{dist}(F_2, x)$. I claim that $B(x, \frac{\varepsilon}{5}) \subseteq \mathcal{O}_1$. Let $y \in B(x, \frac{\varepsilon}{5})$. Then,

$$\begin{aligned}
\text{dist}(F_2, y) &\geq \rho(y, y') - \frac{\varepsilon}{5} && \text{for some } y' \in F_2 \\
&\geq \rho(x, y') - \rho(x, y) + \frac{\varepsilon}{5} && \text{reverse triangle inequality} \\
&\geq \text{dist}(F_2, x) - \frac{2\varepsilon}{5} \\
&\geq \text{dist}(F_1, x) + \varepsilon - \frac{2\varepsilon}{5} \\
&\geq \rho(x, \tilde{y}) + \frac{2\varepsilon}{5} && \text{for some } \tilde{y} \in F_1 \\
&\geq \rho(y, \tilde{y}) - \rho(y, x) + \frac{2\varepsilon}{5} && \text{reverse triangle inequality} \\
&\geq \rho(y, \tilde{y}) - \frac{\varepsilon}{5} + \frac{2\varepsilon}{5} \\
&\geq \text{dist}(F_1, y) + \frac{\varepsilon}{5} > \text{dist}(F_1, y),
\end{aligned}$$

hence, $y \in \mathcal{O}_1$ and thus \mathcal{O}_1 open. Similar proof follows for \mathcal{O}_2 . ■

↪ **Proposition 1.10:** Let X Tychonoff. Then X normal $\Leftrightarrow \forall F \subseteq X$ closed and neighborhood \mathcal{U} of F , there exists an open set \mathcal{O} such that

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}.$$

This is called the “nested neighborhood property” of normal spaces.

PROOF. (\Rightarrow) Let F closed and \mathcal{U} a neighborhood of F . Then, F and \mathcal{U}^c closed disjoint sets so by normality there exists \mathcal{O}, \mathcal{V} disjoint open neighborhoods of F, \mathcal{U}^c respectively. So, $\mathcal{O} \subseteq \mathcal{V}^c$ hence $\overline{\mathcal{O}} \subseteq \overline{\mathcal{V}^c} = \mathcal{V}^c$ and thus

$$F \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{V}^c \subseteq \mathcal{U}.$$

(\Leftarrow) Let A, B be disjoint closed sets. Then, B^c open and moreover $A \subseteq B^c$. Hence, there exists some open set \mathcal{O} such that $A \subseteq \mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq B^c$, and thus $B \subseteq \overline{\mathcal{O}}^c$. Then, \mathcal{O} and $\overline{\mathcal{O}}^c$ are disjoint open neighborhoods of A, B respectively so X normal. ■

↪ **Definition 1.22** (Separable): A space X is called *separable* if it contains a countable dense subset.

↪ **Definition 1.23** (1st, 2nd Countable): A topological space (X, \mathcal{T}) is called

- *1st countable* if there is a countable base at each point; and
- *2nd countable* if there is a countable base for all of \mathcal{T} .

⊗ **Example 1.2:** Every metric space is first countable; for $x \in X$ let $\mathcal{B}_x = \{B(x, \frac{1}{n}) \mid n \in \mathbb{N}\}$.

↪ **Proposition 1.11:** Every 2nd countable space is separable.

↪ **Definition 1.24** (Convergence): Let $\{x_n\} \subseteq X$. Then, we say $x_n \rightarrow x$ in \mathcal{T} if for every neighborhood \mathcal{U}_x , there exists an N such that $\forall n \geq N, x_n \in \mathcal{U}_x$.

Remark 1.11: In general spaces, such a limit may not be unique. For instance, under the trivial topology, the only nonempty neighborhood is the whole space, so every sequence converges to every point in the space.

↪ **Proposition 1.12:** Let (X, \mathcal{T}) be Hausdorff. Then, all limits are unique.

PROOF. Suppose otherwise, that $x_n \rightarrow$ both x and y . If $x \neq y$, then since X Hausdorff there are disjoint neighborhoods $\mathcal{U}_x, \mathcal{U}_y$ containing x, y . But then x_n cannot be on both \mathcal{U}_x and \mathcal{U}_y for sufficiently large n , contradiction. ■

↪ **Proposition 1.13:** Let X be 1st countable and $E \subseteq X$. Then, $x \in \overline{E} \Leftrightarrow$ there exists $\{x_j\} \subseteq E$ such that $x_j \rightarrow x$.

PROOF. (\Rightarrow) Let $\mathcal{B}_x = \{B_j\}$ be a base for X at $x \in \overline{E}$. Wlog, $B_j \supseteq B_{j+1}$ for every $j \geq 1$ (by replacing with intersections, etc if necessary). Hence, $B_j \cap E \neq \emptyset$ for every j . Let $x_j \in B_j \cap E$, then by the nesting property $x_j \rightarrow x$ in \mathcal{T} .

(\Leftarrow) Suppose otherwise, that $x \notin \overline{E}$. Let $\{x_j\} \in E_j$. Then, \overline{E}^c open, and contains x . Then, \overline{E}^c a neighborhood of x but does not contain any x_j so $x_j \nrightarrow x$. ■

§1.7 Continuity and Compactness

↪ **Definition 1.25:** Let $(X, \mathcal{T}), (Y, \mathcal{S})$ be two topological spaces. Then, a function $f : X \rightarrow Y$ is said to be continuous at x_0 if for every neighborhood \mathcal{O} of $f(x_0)$ there exists a neighborhood $\mathcal{U}(x_0)$ such that $f(\mathcal{U}) \subseteq \mathcal{O}$. We say f continuous on X if it is continuous at every point in X .

↪ **Proposition 1.14:** f continuous $\Leftrightarrow \forall \mathcal{O}$ open in $Y, f^{-1}(\mathcal{O})$ open in X .

↪ **Definition 1.26** (Weak Topology): Consider $\mathcal{F} := \{f_\lambda : X \rightarrow X_\lambda\}_{\lambda \in \Lambda}$ where X, X_λ topological spaces. Then, let

$$S := \{f_\lambda^{-1}(\mathcal{O}_\lambda) \mid f_\lambda \in \mathcal{F}, \mathcal{O}_\lambda \in X_\lambda\} \subseteq X.$$

We say that the topology $\mathcal{T}(S)$ generated by S is the *weak topology* for X induced by the family \mathcal{F} .

↪ **Proposition 1.15:** The weak topology is the weakest topology in which each f_λ continuous on X .

⊗ **Example 1.3:** The key example of the weak topology is given by the product topology. Consider $\{X_\lambda\}_{\lambda \in \Lambda}$ a collection of topological spaces. We can define a “natural” topology on the product $X := \prod_{\lambda \in \Lambda} X_\lambda$ by consider the weak topology induced by the family of projection maps, namely, if $\pi_\lambda : X \rightarrow X_\lambda$ a coordinate-wise projection and $\mathcal{F} = \{\pi_\lambda : \lambda \in \Lambda\}$, then we say the weak topology induced by \mathcal{F} is the *product topology* on X . In particular, a base for this topology is given, by previous discussions,

$$\mathcal{B} = \left\{ \bigcap_{j=1}^n \pi_{\lambda_j}^{-1}(\mathcal{O}_j) \right\} = \left\{ \prod_{\lambda \in \Lambda} \mathcal{U}_\lambda : \mathcal{U}_\lambda \text{ open and all but finitely many } \mathcal{U}_\lambda = X_\lambda \right\}.$$

↪ **Definition 1.27** (Compactness): A space X is said to be *compact* if every open cover of X admits a finite subcover.

↪ **Proposition 1.16:**

- Closed subsets of compact spaces are compact
- X compact \Leftrightarrow if $\{F_k\} \subseteq X$ -nested and closed, $\bigcap_{k=1}^\infty F_k \neq \emptyset$.
- Continuous images of compact sets are compact
- Continuous real-valued functions on a compact topological space achieve their min, max.

↪ **Proposition 1.17:** Let K compact be contained in a Hausdorff space X . Then, K closed in X .

PROOF. We show K^c open. Let $y \in K^c$. Then for every $x \in K$, there exists disjoint open sets $\mathcal{U}_{xy}, \mathcal{O}_{xy}$ containing y, x respectively. Then, it follows that $\{\mathcal{O}_{xy}\}_{x \in K}$ an open cover of K , and since K compact there must exist some finite subcover, $K \subseteq \bigcup_{i=1}^N \mathcal{O}_{x_i y}$. Let $E := \bigcap_{i=1}^N \mathcal{U}_{x_i y}$. Then, E is an open neighborhood of y with $E \cap \mathcal{O}_{x_i y} = \emptyset$ for every

$i = 1, \dots, N$. Thus, $E \subseteq \bigcap_{i=1}^N \mathcal{O}_{x_i y}^c = \left(\bigcup_{i=1}^N \mathcal{O}_{x_i y} \right)^c \subseteq K^c$ so since y was arbitrary K^c open. ■

↪ **Definition 1.28** (Sequential Compactness): We say (X, \mathcal{T}) *sequentially compact* if every sequence in X has a converging subsequence with limit contained in X .

↪ **Proposition 1.18**: Let (X, \mathcal{T}) second countable. Then, X compact \Leftrightarrow sequentially compact.

PROOF. (\Rightarrow) Let $\{x_k\} \subseteq X$ and put $F_n := \overline{\{x_k \mid k \geq n\}}$. Then, $\{F_n\}$ defines a sequence of closed and nested subsets of X and, since X compact, $\bigcap_{n=1}^{\infty} F_n$ nonempty. Let x_0 in this intersection. Since X 2nd and so in particular 1st countable, let $\{B_j\}$ a (wlog nested) countable base at x_0 . $x_0 \in F_n$ for every $n \geq 1$ so each B_j must intersect some F_n . Let n_j be an index such that $x_{n_j} \in B_j$. Then, if \mathcal{U} a neighborhood of x_0 , there exists some N such that $B_j \subseteq \mathcal{U}$ for every $j \geq N$ and thus $\{x_{n_j}\} \subseteq B_N \subseteq \mathcal{U}$, so $x_{n_j} \rightarrow x_0$ in X .

(\Leftarrow) Remark that since X second countable, every open cover of X certainly has a countable subcover by intersecting a given cover with our countable basis. So, assume we have a countable cover $X \subseteq \bigcup_{n=1}^{\infty} \mathcal{O}_n$ and suppose towards a contradiction that no finite subcover exists. Then, for every $n \geq 1$, there exists some $m(n) \geq n$ such that $\mathcal{O}_{m(n)} \setminus \bigcup_{i=1}^n \mathcal{O}_i \neq \emptyset$. Let x_n in this set for every $n \geq 1$. Since X sequentially compact, there exists a convergent subsequence $\{x_{n_k}\} \subseteq \{x_n\}$ such that $x_{n_k} \rightarrow x_0$ in X , so there exists some \mathcal{O}_N such that $x_0 \in \mathcal{O}_N$. But by construction, $x_{n_k} \notin \mathcal{O}_N$ if $n_k \geq N$, and we have a contradiction. ■

↪ **Theorem 1.8**: If X compact and Hausdorff, X normal.

PROOF. We show that any closed set F and any point $x \notin F$ can be separated by disjoint open sets. Then, the proof in the more general case follows.

For each $y \in X$, X is Hausdorff so there exists disjoint open neighborhoods \mathcal{O}_{xy} and \mathcal{U}_{xy} of x, y respectively. Then, $\{\mathcal{U}_{xy} \mid y \in F\}$ defines an open cover of F . Since F closed and thus, being a subset of a compact space, compact, there exists a finite subcover $F \subseteq \bigcup_{i=1}^N \mathcal{U}_{xy_i}$. Put $\mathcal{N} := \bigcap_{i=1}^N \mathcal{O}_{xy_i}$. This is an open set containing x , with $\mathcal{N} \cap \bigcup_{i=1}^N \mathcal{U}_{xy_i} = \emptyset$ hence F and x separated by $\mathcal{N}, \bigcup_{i=1}^N \mathcal{U}_{xy_i}$. ■

§1.8 Connected Topological Spaces

↪ **Definition 1.29** (Separate): 2 non-empty sets $\mathcal{O}_1, \mathcal{O}_2$ *separate* X if $\mathcal{O}_1, \mathcal{O}_2$ disjoint and $X = \mathcal{O}_1 \cup \mathcal{O}_2$.

↪ **Definition 1.30** (Connected): We say X *connected* if it cannot be separated.

Remark 1.12: Note that if X can be separated, then $\mathcal{O}_1, \mathcal{O}_2$ are closed as well as open, being complements of each other.

↪ **Proposition 1.19:** Let $f : X \rightarrow Y$ continuous. Then, if X connected, so is $f(X)$.

PROOF. Suppose otherwise, that $f(X) = \mathcal{O}_1 \sqcup \mathcal{O}_2$ for nonempty, open, disjoint $\mathcal{O}_1, \mathcal{O}_2$. Then, $X = f^{-1}(\mathcal{O}_1) \sqcup f^{-1}(\mathcal{O}_2)$, and each of these inverse images remain nonempty and open in X , so this a contradiction to the connectedness of X . ■

Remark 1.13: On \mathbb{R} , $C \subseteq \mathbb{R}$ connected \Leftrightarrow an interval \Leftrightarrow convex.

↪ **Definition 1.31** (Intermediate Value Property): We say X has the intermediate value property (IVP) if $\forall f \in C(X)$, $f(X)$ an interval.

↪ **Proposition 1.20:** X has IVP $\Leftrightarrow X$ connected.

PROOF. (\Leftarrow) If X connected, $f(X)$ connected in \mathbb{R} hence an interval.

(\Rightarrow) Suppose otherwise, that $X = \mathcal{O}_1 \sqcup \mathcal{O}_2$. Then define the function $f : X \rightarrow \mathbb{R}$ by $x \mapsto \begin{cases} 1 & \text{if } x \in \mathcal{O}_2 \\ 0 & \text{if } x \in \mathcal{O}_1 \end{cases}$. Then, for every $A \subseteq \mathbb{R}$,

$$f^{-1}(A) = \begin{cases} \emptyset & \text{if } \{0, 1\} \not\subseteq A \\ \mathcal{O}_1 & \text{if } 0 \in A \\ \mathcal{O}_2 & \text{if } 1 \in A \\ X & \text{if } \{0, 1\} \subseteq A \end{cases},$$

which are all open sets, hence f continuous. But $f(X) = \{0, 1\}$ which is not an interval, hence the IVP fails and so X must be connected. ■

↪ **Definition 1.32** (Arcwise/Path Connected): X arc connected/path connected if $\forall x, y \in X$, there exists a continuous function $f : [0, 1] \rightarrow X$ such that $f(0) = x, f(1) = y$.

↪ **Proposition 1.21:** Arc connected \Rightarrow connected.

PROOF. Suppose otherwise, $X = \mathcal{O}_1 \sqcup \mathcal{O}_2$. Let $x \in \mathcal{O}_1, y \in \mathcal{O}_2$ and define a continuous function $f : [0, 1] \rightarrow X$ such that $f(0) = x$ and $f(1) = y$. Then, $f^{-1}(\mathcal{O}_i)$ each open, nonempty and disjoint for $i = 1, 2$, but

$$f^{-1}(\mathcal{O}_1) \sqcup f^{-1}(\mathcal{O}_2) = [0, 1],$$

a contradiction to the connectedness of $[0, 1]$. ■

§1.9 Urysohn's Lemma and Urysohn's Metrization Theorem

We present the main lemma of this section first, but need more tools before proving it.

↪ **Lemma 1.2** (Urysohn's): Let $A, B \subseteq X$ closed and disjoint subsets of a normal space X . Then, $\forall [a, b] \subseteq \mathbb{R}$, there exists a continuous function $f : [a, b] \rightarrow \mathbb{R}$ such that $f(X) \subseteq [a, b]$, $f|_A = a$ and $f|_B = b$.

Remark 1.14: We have a partial converse of this statement as well:

↪ **Proposition 1.22:** Let X Tychonoff and suppose X satisfies the properties of Urysohn's Lemma. Then, X normal.

PROOF. Let A, B be closed nonempty disjoint subsets. Let $f : X \rightarrow \mathbb{R}$ continuous such that $f|_A = 0$, $f|_B = 1$ and $0 \leq f \leq 1$. Let I_1, I_2 be two disjoint open intervals in \mathbb{R} with $0 \in I_1$ and $1 \in I_2$. Then, $f^{-1}(I_1)$ open and contains A , and $f^{-1}(I_2)$ open and contains B . Moreover, $f^{-1}(I_1) \cap f^{-1}(I_2) = \emptyset$; hence, $f^{-1}(I_1), f^{-1}(I_2)$ disjoint open neighborhoods of A, B respectively, so indeed X normal. ■

↪ **Definition 1.33** (Normally Ascending): Let (X, \mathcal{T}) a topological space and $\Lambda \subseteq \mathbb{R}$. A collection of open sets $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ is said to be *normally ascending* if $\forall \lambda_1, \lambda_2 \in \Lambda$,

$$\overline{\mathcal{O}_{\lambda_1}} \subseteq \mathcal{O}_{\lambda_2} \text{ if } \lambda_1 < \lambda_2.$$

↪ **Lemma 1.3:** Let $\Lambda \subseteq (a, b)$ a dense subset, and let $\{\mathcal{O}_\lambda\}_{\lambda \in \Lambda}$ a normally ascending collection of subsets of X . Let $f : X \rightarrow \mathbb{R}$ defined such that

$$f(x) = \begin{cases} b & \text{if } x \in \left(\bigcup_{\lambda \in \Lambda} \mathcal{O}_\lambda\right)^c \\ \inf\{\lambda \in \Lambda \mid x \in \mathcal{O}_\lambda\} & \text{else} \end{cases}.$$

Then, f continuous.

PROOF. We claim $f^{-1}(-\infty, c)$ and $f^{-1}(c, \infty)$ open for every $c \in \mathbb{R}$. Since such sets define a subbase for \mathbb{R} , it suffices to prove continuity on these sets. We show just the first for convenience. Notice that since $f(x) \in [a, b]$, if $c \in (a, b)$ then $f^{-1}(-\infty, c) = f^{-1}[a, c)$, so really it suffices to show that $f^{-1}[a, c)$ open to complete the proof.

Suppose $x \in f^{-1}([a, c])$ so $a \leq f(x) < c$. Let $\lambda \in \Lambda$ be such that $a < \lambda < f(x)$. Then, $x \notin \mathcal{O}_\lambda$. Let also $\lambda' \in \Lambda$ such that $f(x) < \lambda' < c$. By density of Λ , there exists a $\varepsilon > 0$ such that $f(x) + \varepsilon \in \Lambda$, so in particular

$$\overline{\mathcal{O}_{f(x)+\varepsilon}} \subseteq \mathcal{O}_{\lambda'} \Rightarrow x \in \mathcal{O}_{\lambda'},$$

by nesting. So, repeating this procedure, we find

$$f^{-1}([a, c)) \subseteq \bigcup_{a \leq \lambda < \lambda' < c} \mathcal{O}_{\lambda'} \setminus \overline{\mathcal{O}}_{\lambda},$$

noticing the set on the right is open. By similar reasoning, the opposite inclusion holds and we have equality. Hence, f continuous. \blacksquare

Lemma 1.4: Let X normal, $F \subseteq X$ closed, and \mathcal{U} a neighborhood of F . Then, for any $(a, b) \subseteq \mathbb{R}$, there exists a dense subset $\Lambda \subseteq (a, b)$ and a normally ascending collection $\{\mathcal{O}_{\lambda}\}_{\lambda \in \Lambda}$ such that

$$F \subseteq \mathcal{O}_{\lambda} \subseteq \overline{\mathcal{O}}_{\lambda} \subseteq \mathcal{U}, \quad \forall \lambda \in \Lambda.$$

Remark 1.15: This is essentially a generalization of the nested neighborhood property, and indeed the proof essentially just uses this property repeatedly to construct the collection $\{\mathcal{O}_{\lambda}\}$.

PROOF. Without loss of generality, we assume $(a, b) = (0, 1)$, for the two intervals are homeomorphic, i.e. the function $f : (0, 1) \rightarrow \mathbb{R}$, $f(x) := a(1 - x) + bx$ is continuous, invertible with continuous inverse and with $f(0) = a$, $f(1) = b$ so a homeomorphism.

Let

$$\Lambda := \left\{ \frac{m}{2^n} \mid m, n \in \mathbb{N} \mid 1 \leq m \leq 2^{n-1} \right\} = \bigcup_{n \in \mathbb{N}} \underbrace{\left\{ \frac{m}{2^n} \mid m \in \mathbb{N}, 1 \leq m \leq 2^{n-1} \right\}}_{=: \Lambda_n},$$

which is clearly dense in $(0, 1)$. We need now to define our normally ascending collection. We do so by defining on each Λ_1 and proceeding inductively.

For Λ_1 , since X normal, let $\mathcal{O}_{1/2}$ be such that $F \subseteq \mathcal{O}_{1/2} \subseteq \overline{\mathcal{O}}_{1/2} \subseteq \mathcal{U}$, which exists by the nested neighborhood property.

For $\Lambda_2 = \{\frac{1}{4}, \frac{3}{4}\}$, we use the nested neighborhood property again, but first with F as the closed set and $\mathcal{O}_{1/2}$ an open neighborhood of it, and then with $\overline{\mathcal{O}}_{1/2}$ as the closed set and \mathcal{U} an open neighborhood of it. In this way, we find

$$\underbrace{F \subseteq \mathcal{O}_{1/4} \subseteq \overline{\mathcal{O}}_{1/4} \subseteq \mathcal{O}_{1/2}}_{\text{nested nbhd}} \subseteq \overbrace{\overline{\mathcal{O}}_{1/2} \subseteq \mathcal{O}_{3/4} \subseteq \overline{\mathcal{O}}_{3/4}}^{\text{nested nbhd}} \subseteq \mathcal{U}.$$

We repeat in this manner over all of Λ , in the end defining a normally ascending collection $\{\mathcal{O}_{\lambda}\}_{\lambda \in \Lambda}$. \blacksquare

PROOF (Of Urysohn's Lemma, [Lem. 1.2](#)). Let $F = A$ and $\mathcal{U} = B^c$ as in the previous lemma [Lem. 1.4](#). Then, there is some dense subset $\Lambda \subseteq (a, b)$ and a normally ascending collection $\{\mathcal{O}_{\lambda}\}_{\lambda \in \Lambda}$ such that $A \subseteq \mathcal{O}_{\lambda} \subseteq \overline{\mathcal{O}}_{\lambda} \subseteq B^c$ for every $\lambda \in \Lambda$. Let $f(x)$ as in the previous lemma, [Lem. 1.3](#). Then, if $x \in B$, $B \subseteq \left(\bigcup_{\lambda \in \Lambda} \mathcal{O}_{\lambda} \right)^c$ and so $f(x) = b$.

Otherwise if $x \in A$, then $x \in \bigcap_{\lambda \in \Lambda} \mathcal{O}_\lambda$ and thus $f(x) = \inf\{\lambda \in \Lambda\} = a$. By the first lemma, f continuous, so we are done. ■

↪ **Theorem 1.9** (Urysohn's Metrization Theorem): Let X be a second countable topological space. Then, X is metrizable (that is, there exists a metric on X that induces the topology) if and only if X normal.

PROOF. (\Rightarrow) We have already showed, every metric space is normal.

(\Leftarrow) Let $\{\mathcal{U}_n\}$ be a countable basis for \mathcal{T} and put

$$A := \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid \overline{\mathcal{U}_n} \subseteq \mathcal{U}_m\}.$$

By Urysohn's lemma, for each $(n, m) \in A$ there is some continuous function $f_{n,m} : X \rightarrow \mathbb{R}$ such that $f_{n,m}$ is 1 on \mathcal{U}_m^c and 0 on $\overline{\mathcal{U}_n}$ (these are disjoint closed sets). For $x, y \in X$, define

$$\rho(x, y) := \sum_{(n,m) \in A} \frac{1}{2^{n+m}} |f_{n,m}(x) - f_{n,m}(y)|.$$

The absolute valued term is ≤ 2 , so this function will always be finite. Moreover, one can verify that it is indeed a metric on X . It remains to show that it induces the same topology; it suffices to compare bases of the two.

Let $x \in \mathcal{U}_m$. We wish to show there exists $B_\rho(x, \varepsilon) \subseteq \mathcal{U}_m$. $\{x\}$ is closed in X being normal, so there exists some n such that

$$\{x\} \subseteq \mathcal{U}_n \subseteq \overline{\mathcal{U}_n} \subseteq \mathcal{U}_m,$$

so $(n, m) \in A$ and so $f_{n,m}(x) = 0$. Let $\varepsilon = \frac{1}{2^{n+m}}$. Then, if $\rho(x, y) < \varepsilon$, it must be

$$\begin{aligned} \frac{1}{2^{n+m}} &> \sum_{(n',m') \in A} \frac{1}{2^{n'+m'}} |f_{n',m'}(x) - f_{n',m'}(y)| \\ &\geq \frac{1}{2^{n+m}} \underbrace{|f_{n,m}(x) - f_{n,m}(y)|}_{=0} \\ &= \frac{1}{2^{n+m}} |f_{n,m}(y)|, \end{aligned}$$

so $|f_{n,m}(y)| < 1$ and thus $y \notin \mathcal{U}_m^c$ so $y \in \mathcal{U}_m$. It follow that $B_\rho(x, \varepsilon) \subseteq \mathcal{U}_m$, and so every open set in X is open with respect to the metric topology.

Conversely, if $B_\rho(x, \varepsilon)$ some open ball in the metric topology, then notice that $y \mapsto \rho(x, y)$ for fixed x a continuous function, and thus $(\rho(x, \cdot))^{-1}(-\varepsilon, \varepsilon)$ an open set in \mathcal{T} containing x . But this set also just equal to $B_\rho(x, \varepsilon)$, hence $B_\rho(x, \varepsilon)$ open in \mathcal{T} . We conclude the two topologies are equal, completing the proof. ■

Remark 1.16: Recall metric \Rightarrow first countable hence not first countable \Rightarrow not metrizable.

§1.10 Stone-Weierstrass Theorem

We need to use the following theorem, which we'll prove later.

↪ **Theorem 1.10** (Weierstrass Approximation Theorem): Let $f : [a, b] \rightarrow \mathbb{R}$ continuous. Then, for every $\varepsilon > 0$, there exists a polynomial $p(x)$ such that $\|f - p\|_\infty < \varepsilon$.

↪ **Definition 1.34** (Algebra, Separation of Points): We call a subset $\mathcal{A} \subseteq C(X)$ an *algebra* if it is a linear subspace that is closed under multiplication (that is, $f, g \in \mathcal{A} \Rightarrow f \cdot g \in \mathcal{A}$).

We say \mathcal{A} *separates points* in X if for every $x, y \in X$, there exists an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

↪ **Theorem 1.11** (Stone-Weierstrass): Let X be a compact Hausdorff space. Suppose $\mathcal{A} \subseteq C(X)$ an algebra that separates points and contains constant functions. Then, \mathcal{A} dense in $C(X)$.

We tacitly assume the conditions of the theorem in the following lemmas as not to restate them.

↪ **Lemma 1.5**: For every $F \subseteq X$ closed, and every $x_0 \in F^c$, there exists a neighborhood $\mathcal{U}(x_0)$ such that $F \cap \mathcal{U} = \emptyset$ and $\forall \varepsilon > 0$ there is some $h \in \mathcal{A}$ such that $h < \varepsilon$ on \mathcal{U} , $h > 1 - \varepsilon$ on F , and $0 \leq h \leq 1$ on X .

In particular, \mathcal{U} is *independent* of choice of ε .

PROOF. Our first claim is that for every $y \in F$, there is a $g_y \in \mathcal{A}$ such that $g_y(x_0) = 0$ and $g_y(y) > 0$, and moreover $0 \leq g_y \leq 1$. Since \mathcal{A} separates points, there is an $f \in \mathcal{A}$ such that $f(x_0) \neq f(y)$. Then, let

$$g_y(x) := \left[\frac{f(x) - f(x_0)}{\|f - f(x_0)\|_\infty} \right]^2.$$

Then, every operation used in this new function keeps $g_y \in \mathcal{A}$. Moreover one readily verifies it satisfies the desired qualities. In particular since g_y continuous, there is a neighborhood \mathcal{O}_y such that $g_y|_{\mathcal{O}_y} > 0$. Hence, we know that $F \subseteq \bigcup_{y \in F} \mathcal{O}_y$, but F closed and so compact, hence there exists a finite subcover i.e. some $n \geq 1$ and finite sequence $\{y_i\}_{i=1}^n$ such that $F \subseteq \bigcup_{i=1}^n \mathcal{O}_{y_i}$. Let for each y_i $g_{y_i} \in \mathcal{A}$ with the properties from above, and consider the “averaged” function

$$g(x) := \frac{1}{n} \sum_{i=1}^n g_{y_i}(x) \in \mathcal{A}.$$

Then, $g(x_0) = 0$, $g > 0$ on F and $0 \leq g \leq 1$ on all of X . Hence, there is some $1 > c > 0$ such that $g \geq c$ on F , and since g continuous at x_0 there exists some $\mathcal{U}(x_0)$ such that $g < \frac{c}{2}$ on \mathcal{U} , with $\mathcal{U} \cap F = \emptyset$. So, $0 \leq g|_{\mathcal{U}} < \frac{c}{2}$, and $1 \geq g|_F \geq c$. To complete the proof, we need $(0, \frac{c}{2}) \leftrightarrow (0, \varepsilon)$ and $(c, 1) \leftrightarrow (1 - \varepsilon, 1)$. By the Weierstrass Approximation Theorem, there exists some polynomial p such that $p|_{[0, \frac{c}{2}]} < \varepsilon$ and $p|_{[c, 1]} > 1 - \varepsilon$. Then if we let $h(x) := (p \circ g)(x)$, this is just a polynomial of g hence remains in \mathcal{A} , and we find

$$h|_{\mathcal{U}} < \varepsilon, \quad h|_F > 1 - \varepsilon, \quad 0 \leq h \leq 1.$$

■

↪ **Lemma 1.6:** For every disjoint closed set A, B and $\varepsilon > 0$, there exists $h \in \mathcal{A}$ such that $h|_A < \varepsilon$, $h|_B > 1 - \varepsilon$, and $0 \leq h \leq 1$ on X .

PROOF. Let $F = B$ as in the last lemma. Let $x \in A$, then there exists $\mathcal{U}_x \cap B = \emptyset$ and for every $\varepsilon > 0$, $h|_{\mathcal{U}_x} < \varepsilon$ and $h|_B > 1 - \varepsilon$ and $0 \leq h \leq 1$. Then $A \subseteq \bigcup_{x \in A} \mathcal{U}_x$. Since A closed so compact, $A \subseteq \bigcup_{i=1}^N \mathcal{U}_{x_i}$. Let $\varepsilon_0 < \varepsilon$ such that $(1 - \frac{\varepsilon_0}{N})^N > 1 - \varepsilon$. For each i , let $h_i \in \mathcal{A}$ such that $h_i|_{\mathcal{U}_{x_i}} < \frac{\varepsilon_0}{N}$, $h_i|_B > 1 - \frac{\varepsilon_0}{N}$ and $0 \leq h_i \leq 1$. Then, put

$$h(x) = h_1(x) \cdot h_2(x) \cdots h_N(x) \in \mathcal{A}.$$

Then, $0 \leq h \leq 1$ and $h|_B > (1 - \frac{\varepsilon_0}{N})^N > 1 - \varepsilon$. Then, for every $x \in A$, $x \in \mathcal{U}_{x_i}$ so $h_i(x) < \frac{\varepsilon_0}{N}$ and $h_i(x) \leq i$ so $h(x) < \frac{\varepsilon_0}{N}$ so $h|_A < \frac{\varepsilon_0}{N} < \varepsilon$. ■

PROOF. (Of Stone-Weierstrass) WLOG, assume $f \in C(X)$, $0 \leq f \leq 1$, by replacing with

$$\tilde{f}(x) = \frac{f(x) + \|f\|_{\infty}}{\|f\|_{\infty} + \|f\|_{\infty}}$$

if necessary, since if there exists a $\tilde{g} \in \mathcal{A}$ such that $\|\tilde{f} - \tilde{g}\|_{\infty} < \varepsilon$, then using the properties of \mathcal{A} we can find some appropriate $g \in \mathcal{A}$ such that $\|f - g\|_{\infty} < \varepsilon$.

Fix $n \in \mathbb{N}$, and consider the set $\{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$, and let for $1 \leq j \leq n$

$$A_j := \left\{ x \in X \mid f(x) \leq \frac{j-1}{n} \right\}, \quad B_j := \left\{ x \in X \mid f(x) \geq \frac{j}{n} \right\},$$

which are both closed and disjoint. By the lemma, there exists $g_j \in \mathcal{A}$ such that

$$g_j|_{A_j} < \frac{1}{n}, \quad g_j|_{B_j} > 1 - \frac{1}{n},$$

with $0 \leq g_j \leq 1$. Let then

$$g(x) := \frac{1}{n} \sum_{j=1}^n g_j(x) \in \mathcal{A}.$$

We claim then $\|f - g\|_{\infty} \leq \frac{3}{n}$, which proves the claim by taking n sufficiently large.

Suppose $k \in [1, n]$. If $f(x) \leq \frac{k}{n}$, then

$$g_j(x) = \begin{cases} < \frac{1}{n} & \text{if } j-1 \geq k, \\ \leq 1 & \text{else} \end{cases},$$

so

$$g(x) = \frac{1}{n} \sum_{j=1}^n g_j(x) = \frac{1}{n} \left[\sum_{j=1}^k g_j(x) + \sum_{j=k+1}^n g_j(x) \right] \leq \frac{1}{n} \left[k + \frac{n-k}{n} \right] \leq \frac{k}{n} + \frac{n-k}{n^2} \leq \frac{k+1}{n}.$$

Similarly if $f(x) \geq \frac{k-1}{n}$, then

$$g_j(x) = \begin{cases} > 1 - \frac{1}{n} & \text{if } j \leq k-1, \\ \geq 0 & \text{else} \end{cases},$$

so

$$g(x) \geq \frac{1}{n} \sum_{j=1}^{k-1} \left(1 - \frac{1}{n} \right) \geq \frac{1}{n} (k-1) \left(1 - \frac{1}{n} \right) = \frac{k-1}{n} - \frac{k-1}{n^2} \geq \frac{k-2}{n}.$$

So, we've show that if $\frac{k-1}{n} \leq f(x) \leq \frac{k}{n}$, then $\frac{k-2}{n} \leq g(x) \leq \frac{k+1}{n}$, and so repeating this argument and applying triangle inequality we conclude $\|f - g\|_\infty \leq \frac{3}{n}$. ■

↪ **Theorem 1.12** (Borsuk): X compact, Hausdorff and $C(X)$ separable $\Leftrightarrow X$ is metrizable.

§2 FUNCTIONAL ANALYSIS

Here, we will primarily work with a normed vector space (nvs). Moreover, we usually work in:

↪ **Definition 2.1** (Banach Space): A normed vector space $(X, \|\cdot\|)$ is a *Banach space* if it is complete as a metric space under the norm-induced metric.

§2.1 Introduction to Linear Operators

↪ **Definition 2.2** (Linear Operator, Operator Norm): Let X, Y be vector spaces. Then, a map $T : X \rightarrow Y$ is called *linear* if $\forall x, y \in X, \alpha, \beta \in \mathbb{R}, T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$.

If X, Y normed vector spaces, we say T is a bounded linear operator if T linear and the *operator norm*

$$\|T\| = \|T\|_{\mathcal{L}(X, Y)} = \sup_{\substack{x \in X, \\ \|x\|_X \leq 1}} \|Tx\|_Y < \infty$$

is finite. Then, we put

$$\mathcal{L}(X, Y) := \{\text{bounded linear operators } X \rightarrow Y\}.$$

We'll also write $\mathcal{L}(X) := \mathcal{L}(X, X)$.

↪ **Theorem 2.1** (Bounded iff Continuous): If X, Y are nvs, $T \in \mathcal{L}(X, Y)$ iff and only if T is continuous, i.e. if $x_n \rightarrow x$ in X , then $Tx_n \rightarrow Tx$ in Y .

PROOF. If $T \in \mathcal{L}(X, Y)$,

$$\begin{aligned}\|Tx_n - Tx\|_Y &= \|T(x_n - x)\|_Y \\ &= \|x_n - x\|_X \cdot \left\| \frac{T(x_n - x)}{\|x_n - x\|_X} \right\|_Y \\ &\leq \underbrace{\|T\|}_{< \infty} \|x_n - x\|_X \rightarrow 0,\end{aligned}$$

hence T continuous. Conversely, if T continuous, then by linearity $T0 = 0$, so by continuity, there is some $\delta > 0$ such that $\|Tx\|_Y < 1$ if $\|x\|_X < \delta$. For $x \in X$ nonzero, let $\lambda = \frac{\delta}{\|x\|_X}$. Then, $\|\lambda x\|_X \leq \delta$ so $\|T(\lambda x)\|_Y < 1$, i.e. $\frac{\|T(x)\|_Y \delta}{\|x\|_X} < 1$. Hence,

$$\|T\| = \sup_{x \in X: x \neq 0} \frac{\|T(x)\|_Y}{\|x\|_X} \leq \frac{1}{\delta},$$

so $T \in \mathcal{L}(X, Y)$. ■

↪ **Proposition 2.1** (Properties of $\mathcal{L}(X, Y)$): If X, Y nvs, $\mathcal{L}(X, Y)$ a nvs, and if X, Y Banach, then so is $\mathcal{L}(X, Y)$.

PROOF. (a) For $T, S \in \mathcal{L}(X, Y)$, $\alpha, \beta \in \mathbb{R}$, and $x \in X$, then

$$\begin{aligned}\|(\alpha T + \beta S)(x)\|_Y &\leq |\alpha| \|Tx\|_Y + |\beta| \|Sx\|_Y \\ &\leq |\alpha| \|T\| \|x\|_X + |\beta| \|S\| \|x\|_X.\end{aligned}$$

Dividing both sides by $\|x\|$, we find $\|\alpha T + \beta S\| < \infty$. The same argument gives the triangle inequality on $\|\cdot\|$. Finally, $T = 0$ iff $\|Tx\|_Y = 0$ for every $x \in X$ iff $\|T\| = 0$.

(b) Let $\{T_n\} \subseteq \mathcal{L}(X, Y)$ be a Cauchy sequence. We have that

$$\|T_n x - T_m x\|_Y \leq \|T_n - T_m\| \|x\|_X,$$

so in particular the sequence $\{T_n(x)\}$ a Cauchy sequence in Y for any $x \in X$. Y complete so this sequence converges, say $T_n(x) \rightarrow y^*$ in Y . Let $T(x) := y^*$ for each x . We claim that $T \in \mathcal{L}(X, Y)$ and that $T_n \rightarrow T$ in the operator norm. We check:

$$\begin{aligned}\alpha T(x_1) + \beta T(x_2) &= \lim_{n \rightarrow \infty} \alpha T_n(x_1) + \lim_{n \rightarrow \infty} \beta T_n(x_2) \\ &= \lim_{n \rightarrow \infty} [T_n(\alpha x_1) + T_n(\beta x_2)] \\ &= \lim_{n \rightarrow \infty} T_n(\alpha x_1 + \beta x_2) \\ &= T(\alpha x_1 + \beta x_2),\end{aligned}$$

so T linear.

Let now $\varepsilon > 0$ and N such that for every $n \geq N$ and $k \geq 1$ such that $\|T_n - T_{n+k}\| < \frac{\varepsilon}{2}$. Then,

$$\begin{aligned}\|T_n(x) - T_{n+k}(x)\|_Y &= \|(T_n - T_{n+k})(x)\|_Y \\ &\leq \|T_n - T_{n+k}\| \|x\|_X \\ &< \frac{\varepsilon}{2} \|x\|_X.\end{aligned}$$

Letting $k \rightarrow \infty$, we find that

$$\|T_n(x) - T(x)\|_Y < \frac{\varepsilon}{2} \|x\|_X,$$

so normalizing both sides by $\|x\|_X$, we find $\|T_n - T\| < \frac{\varepsilon}{2}$, and we have convergence. ■

↪ **Definition 2.3** (Isomorphism): We say $T \in \mathcal{L}(X, Y)$ an *isomorphism* if T is bijective and $T^{-1} \in \mathcal{L}(Y, X)$. In this case we write $X \simeq Y$, and say X, Y isomorphic.

§2.2 Finite versus Infinite Dimensional

If X a nvs, then we can look for a basis β such that $\text{span}(\beta) = X$. If $\beta = \{e_1, \dots, e_n\}$ has no proper subset spanning X , then we say $\dim(X) = n$.

As we saw on homework, any two norms on a finite dimensional space are equivalent.

↪ **Corollary 2.1:** (a) Any two nvs of the same finite dimension are isomorphic.

(b) Any finite dimensional space is complete, and so any finite dimensional subspace is closed.

(c) $\overline{B}(0, 1)$ is compact in a finite dimensional space.

PROOF. (a) Let $(X, \|\cdot\|)$ have finite dimension n . Then, we claim $(X, \|\cdot\|) \simeq (\mathbb{R}^n, |\cdot|)$. Let $\{e_1, \dots, e_n\}$ be a basis for X . Let $T : \mathbb{R}^n \rightarrow X$ given by

$$T(x) = \sum_{i=1}^n x_i e_i,$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, which is clearly linear. Moreover,

$$Tx = 0 \Leftrightarrow \sum_{i=1}^n x_i e_i = 0 \Leftrightarrow x = 0,$$

so T injective, and so being linear between two spaces of the same dimension gives T surjective. It remains to check boundedness.

First, we claim $x \mapsto \|T(x)\|$ is a norm on \mathbb{R}^n . $\|T(x)\| = 0 \Leftrightarrow x = 0$ by the injectivity of T , and the properties $\|T(\lambda x)\| = |\lambda| \|Tx\|$ and $\|T(x + y)\| \leq \|Tx\| + \|Ty\|$ follow from linearity of T and the fact that $\|\cdot\|$ already a norm. Hence, $\|T(\cdot)\|$ a norm on \mathbb{R}^n and so equivalent to $|\cdot|$, i.e. there exists constants $C_1, C_2 > 0$ such that

$$C_1|x| \leq \|T(x)\| \leq C_2|x|,$$

for every $x \in X$. It follows that $\|T\|$ (operator norm now) is bounded.

Letting $T(x) = y$, we find similarly

$$C_1'\|y\| \leq |T^{-1}(y)| \leq C_2'\|y\|,$$

so $\|T^{-1}\|$ also bounded. Hence, we've shown any n -dimensional space is isomorphic to \mathbb{R}^n , so by transitivity of isomorphism any two n -dimensional spaces are isomorphic.

(b) The property of completeness is preserved under isomorphism, so this follows from the previous statement since \mathbb{R}^n complete.

(c) Consider $\overline{B}(0, 1) \subseteq X$. Let T be an isomorphism $X \rightarrow \mathbb{R}^n$. Then, for $x \in \overline{B}(0, 1)$, $\|Tx\| \leq \|T\| < \infty$, so $T(\overline{B}(0, 1))$ is a bounded subset of \mathbb{R}^n , and since T and its inverse continuous, $T(\overline{B}(0, 1))$ closed in \mathbb{R}^n . Hence, $T(\overline{B}(0, 1))$ closed and bounded hence compact in \mathbb{R}^n , so since T^{-1} continuous $T^{-1}(T(\overline{B}(0, 1))) = \overline{B}(0, 1)$ also compact, in X . ■

↪ **Theorem 2.2** (Riesz's): If X is an nvs, then $\overline{B}(0, 1)$ is compact if and only if X is finite dimensional.

↪ **Lemma 2.1** (Riesz's): Let $Y \subsetneq X$ be a closed nvs (and X a nvs). Then for every $\varepsilon > 0$, there exists $x_0 \in X$ with $\|x_0\| = 1$ and such that

$$\|x_0 - y\|_X > \varepsilon \quad \forall y \in Y.$$

PROOF. Fix $\varepsilon > 0$. Since $Y \subsetneq X$, let $x \in Y^c$. Y closed so Y^c open and hence there exists some $r > 0$ such that $B(x, r) \cap Y = \emptyset$. In other words,

$$\inf\{\|x - y'\| \mid y' \in Y\} > r > 0.$$

Let then $y_1 \in Y$ be such that

$$r < \|x - y_1\| < \varepsilon^{-1}r,$$

and take

$$x_0 := \frac{x - y_1}{\|x - y_1\|_X}.$$

Then, x_0 a unit vector, and for every $y \in Y$,

$$\begin{aligned} x_0 - y &= \frac{x - y_1}{\|x - y_1\|} - y \\ &= \frac{1}{\|x - y_1\|} [x - y_1 - y \|x - y_1\|] \\ &= \frac{1}{\|x - y_1\|} [x - y'], \end{aligned}$$

where $y' = y_1 + y$ $\|x - y_1\| \in Y$, since it is closed under vector addition. Hence

$$\|x_0 - y\| = \frac{1}{\|x - y_1\|} \|x - y'\| > \frac{\varepsilon}{r} \|x - y'\| > \varepsilon,$$

for every $y \in Y$. ■

PROOF. (Of [Thm. 2.2](#)) (\Leftarrow) By the previous corollary.

(\Rightarrow) Suppose X infinite dimensional. We will show $B := \overline{B}(0, 1)$ not compact.

Claim: there exists $\{x_i\}_{i=1}^{\infty} \subseteq B$ such that $\|x_i - x_j\| > \frac{1}{2}$ if $i \neq j$.

We proceed by induction. Let $x_1 \in B$. Suppose $\{x_1, \dots, x_n\} \subseteq B$ are such that $\|x_i - x_j\| > \frac{1}{2}$. Let $X_n = \text{span}\{x_1, \dots, x_n\}$, so X_n finite dimensional hence $X_n \subsetneq X$. By the previous lemma (taking $\varepsilon = \frac{1}{2}$) there is then some $x_{n+1} \in B$ such that $\|x_1 - x_{n+1}\| > \frac{1}{2}$ for every $i = 1, \dots, n$. We can thus inductively build such a sequence $\{x_i\}_{i=1}^{\infty}$. Then, every subsequence of this sequence cannot be Cauchy so B is not sequentially compact and thus B is not compact. ■

§2.3 Open Mapping and Closed Graph Theorems

\hookrightarrow **Definition 2.4** (T open): If X, Y topological spaces and $T : X \rightarrow Y$ a linear operator, T is said to be *open* if for every $\mathcal{U} \subseteq X$ open, $T(\mathcal{U})$ open in Y .

In particular if X, Y are metric spaces (or nvs), then T is open iff the image of every open ball in X contains an open ball in Y , i.e. $\forall x \in X, r > 0$ there exists $r' > 0$ such that $T(B_X(x, r)) \supseteq B_Y(Tx, r')$. Moreover, by translating/scaling appropriately, it suffices to prove for $x = 0, r = 1$.

\hookrightarrow **Theorem 2.3** (Open Mapping Theorem): Let X, Y be Banach spaces and $T : X \rightarrow Y$ a bounded linear operator. If T is surjective, then T is open.

PROOF. Its enough to show that there is some $r > 0$ such that $T(B_X(0, 1)) \supseteq B_Y(0, r)$.

Claim: $\exists c > 0$ such that $\overline{T(B_X(0, 1))} \supseteq B_Y(0, 2c)$.

Put $E_n = n \cdot \overline{T(B_X(0, 1))}$ for $n \in \mathbb{N}$. Since T surjective, $\bigcup_{n=1}^{\infty} E_n = Y$. Each E_n closed, so by the Baire Category Theorem there exists some index n_0 such that E_{n_0} has nonempty interior, i.e.

$$\text{int}(\overline{T(B_X(0, 1))}) \neq \emptyset,$$

where we drop the index by homogeneity. Pick then $c > 0$ and $y_0 \in Y$ such that $B_Y(y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$. We claim then that $B_Y(-y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$ as well. Indeed, if $B_Y(y_0, 4c) \subseteq \overline{T(B_X(0, 1))}$, then $\forall \tilde{y} \in Y$ with $\|y_0 - \tilde{y}\|_Y < 4c$, Then, $\| -y_0 + \tilde{y}\|_Y < 4c$ so $-\tilde{y} \in B_Y(-y_0, 4c)$. But $\tilde{y} = \lim_{n \rightarrow \infty} T(x_n)$ and so $-\tilde{y} = \lim_{n \rightarrow \infty} T(-x_n)$. Since $\{-x_n\} \subseteq B_X(0, 1)$, this implies $-\tilde{y} \in \overline{T(B_X(0, 1))}$ hence the “subclaim” holds.

Now, for any $\tilde{y} \in B_Y(0, 4c)$, $\|\tilde{y}\| \leq 4c$ so

$$\tilde{y} = y_0 - \underbrace{y_0 + \tilde{y}}_{\in B_Y(-y_0, 4c)} = \overbrace{y_0 + \tilde{y}}^{\in B(y_0, 4c)} - y_0.$$

Therefore,

$$\begin{aligned} B_Y(0, 4c) &= B_Y(y_0 - y_0, 4c) \\ &\subseteq B_Y(y_0, 4c) + B_Y(-y_0, 4c) \\ \overline{T(B_X(0, 1))} + \overline{T(B_X(0, 1))} &= 2\overline{T(B_X(0, 1))}, \end{aligned}$$

(where summation of two sets is the vector addition of all the elements in the sets), hence $B_Y(0, 2c) \subseteq \overline{T(B_X(0, 1))}$.

We claim next that $T(B_X(0, 1)) \supseteq B_Y(0, c)$. Choose $y \in Y$ with $\|y\|_Y < c$. By the first claim, $B_Y(0, c) \subseteq \overline{T(B_X(0, \frac{1}{2}))}$, so for every $\varepsilon > 0$ there is some $z \in X$ with $\|z\|_X < \frac{1}{2}$ and $\|y - Tz\|_Y < \varepsilon$. Let $\varepsilon = \frac{c}{2}$ and $z_1 \in X$ such that $\|z_1\|_X < \frac{1}{2}$ and $\|y - Tz_1\|_Y < \frac{c}{2}$. But the first claim can also be written as $B_Y(0, \frac{c}{2}) \subseteq \overline{T(B_X(0, \frac{1}{4}))}$ so if $\varepsilon = \frac{c}{4}$, let $z_2 \in X$ such that $\|z_2\|_X < \frac{1}{4}$ and $\|(y - Tz_1) - Tz_2\|_Y < \frac{c}{4}$. Continuing in this manner we find that

$$B_Y\left(0, \frac{c}{2^k}\right) \subseteq \overline{T\left(B_X\left(0, \frac{1}{2^{k+1}}\right)\right)},$$

so exists $z_k \in X$ such that $\|z_k\|_X < \frac{1}{2^k}$ and $\|y - T(z_1 + \dots + z_k)\|_Y < \frac{c}{2^k}$. Let $x_n = z_1 + \dots + z_n \in X$. Then $\{x_n\}$ is Cauchy in X , since

$$\|x_n - x_m\|_X \leq \sum_{k=m}^n \|z_k\|_X < \sum_{k=m}^n \frac{1}{2^k} \rightarrow 0.$$

Since X a Banach space, $x_n \rightarrow \bar{x}$ and in particular $\|\bar{x}\| \leq \sum_{k=1}^{\infty} \|z_k\|_X < \sum_{k=1}^{\infty} \frac{1}{2^k} = 1$, so $\bar{x} \in B_X(0, 1)$. Since T bounded it is continuous, so $Tx_n \rightarrow T\bar{x}$, so $y = T\bar{x}$ and thus $B_Y(0, c) \subseteq T(B_X(0, 1))$. ■

↪ **Corollary 2.2:** Let X, Y Banach and $T : X \rightarrow Y$ be bounded, linear and bijective. Then, T^{-1} continuous.

PROOF. Let $\mathcal{U} \subseteq X$ open. Then, $(T^{-1})^{-1}(\mathcal{U}) = T(\mathcal{U})$ is open since T surjective, so T^{-1} continuous. ■

↪ **Corollary 2.3:** Let $(X, \|\cdot\|_1), (X, \|\cdot\|_2)$ be Banach spaces. Suppose there exists $c > 0$ such that $\|x\|_2 \leq C\|x\|_1$ for every $x \in X$. Then, $\|\cdot\|_1, \|\cdot\|_2$ are equivalent.

PROOF. Let T be the identity linear operator and use the previous corollary. ■

↪ **Definition 2.5** (T closed): If X, Y are nvs and T is linear, the *graph* of T is the set

$$G(T) = \{(x, Tx) \mid x \in X\} \subseteq X \times Y.$$

We then say T is *closed* if $G(T)$ closed in $X \times Y$.

Remark 2.1: Since X, Y are nvs, they are metric spaces so first countable, hence closed \leftrightarrow contains all limit points.

In the product topology, a countable base for $X \times Y$ at (x, y) is given by

$$\left\{ B_X\left(x, \frac{1}{n}\right) \times B\left(y, \frac{1}{m}\right) \right\}_{n, m \in \mathbb{N}}.$$

Then, $G(T)$ closed iff $G(T)$ contains all limit points. How can we put a norm on $X \times Y$ that generates this product topology? Let

$$\|(x, y)\|_1 := \|x\|_X + \|y\|_Y.$$

If $(x_n, y_n) \rightarrow (x, y)$ in the product topology, then since Π_1, Π_2 continuous maps, $(x_n, y_n) \rightarrow (x, y)$ in the $\|\cdot\|_1$ topology. On the other hand if $(x_n, y_n) \rightarrow (x, y)$ in the $\|\cdot\|_1$ norm, then

$$\|x_n - x\|_X \leq \|(x_n, y_n) - (x, y)\|_1,$$

hence since the RHS $\rightarrow 0$ so does the LHS and so $x_n \rightarrow x$ in $\|\cdot\|_X$; similar gives $y_n \rightarrow y$ in $\|\cdot\|_Y$. From here it follows that $(x_n, y_n) \rightarrow (x, y)$ in the product topology.

So, to prove $G(T)$ closed, we just need to prove that if $x_n \rightarrow x$ in X and $Tx_n \rightarrow y$, then $y = Tx_n$.

↪ **Theorem 2.4** (Closed Graph Theorem): Let X, Y be Banach spaces and $T : X \rightarrow Y$ linear. Then, T is continuous iff T is closed.

PROOF. (\Rightarrow) Immediate from the above remark.

(\Leftarrow) Consider the function

$$x \mapsto \|x\|_* := \|x\|_X + \|Tx\|_Y.$$

So by the above, T closed implies $(X, \|\cdot\|_*)$ is complete, i.e. if $x_n \rightarrow x$ in $\|\cdot\|_*$ in X iff $x_n \rightarrow x$ in $\|\cdot\|_X$ and $Tx_n \rightarrow Tx$ in $\|\cdot\|_Y$. However, $\|\cdot\|_X \leq \|\cdot\|_*$, hence since $(X, \|\cdot\|_X)$ and $(X, \|\cdot\|_*)$ are Banach spaces, by the corollary, there is some $C > 0$ such that $\|\cdot\|_* \leq C\|\cdot\|_X$. So,

$$\|x\|_X + \|Tx\|_Y \leq C\|x\|_X,$$

so

$$\|Tx\|_Y \leq \|x\|_X + \|Tx\|_Y \leq C\|x\|_X,$$

so T bounded and thus continuous. ■

Remark 2.2: The Closed Graph Theorem simplifies proving continuity of T . It tells us we can assume if $x_n \rightarrow x$, $\{Tx_n\}$ Cauchy so $\exists y$ such that $Tx_n \rightarrow y$ since Y is Banach. So, it suffices to check that $y = Tx$ to check continuity; we don't need to check convergence of Tx_n .

§2.4 Uniform Boundedness Principle

Recall the following consequence of the Baire Category Theorem:

↪ **Theorem 2.5:** Let $\mathcal{F} \subseteq C(X)$ where (X, ρ) a complete metric space. Suppose \mathcal{F} pointwise bounded. Then, there exists a nonempty open set $\mathcal{O} \subseteq X$ such that there is some $M > 0$ such that $|f(x)| \leq M$ for every $x \in \mathcal{O}, f \in \mathcal{F}$.

This leads to the following result:

↪ **Theorem 2.6 (Uniform Boundedness Principle):** Let X a Banach space and Y a nvs. Consider $\mathcal{F} \subseteq \mathcal{L}(X, Y)$. Suppose \mathcal{F} is pointwise bounded, i.e. for every $x \in X$, there is some $M_x > 0$ such that

$$\|Tx\|_Y \leq M_x, \forall T \in \mathcal{F}.$$

Then, \mathcal{F} is uniformly bounded, i.e. $\exists M > 0$ such that

$$\|T\|_Y \leq M, \forall T \in \mathcal{F}.$$

PROOF. For every $T \in \mathcal{F}$, let $f_T : X \rightarrow \mathbb{R}$ be given by

$$f_T(x) = \|Tx\|_Y.$$

Since $T \in \mathcal{L}(X, Y)$, T is continuous, so $x_n \xrightarrow{X} x \Rightarrow Tx_n \xrightarrow{Y} Tx$, hence $\|Tx_n\|_Y \rightarrow \|Tx\|_Y$ so f_T continuous for each T i.e. $f_T \in C(X)$, so $\{f_T\} \subseteq C(X)$ pointwise bounded. So by the previous theorem, there is some ball $B(x_0, r) \subseteq X$ and some $K > 0$ such that $\|Tx\| \leq K$ for every $x \in B(x_0, r)$ and $T \in \mathcal{F}$. Thus, for every $x \in B(0, r)$,

$$\begin{aligned} \|Tx\| &= \|T(x - x_0 + x_0)\| \\ &\leq \left\| T \underbrace{(x - x_0)}_{\in B(x_0, r)} \right\| + \|Tx_0\| \\ &\leq K + M_{x_0}, \quad \forall x \in B(0, r), T \in \mathcal{F}. \end{aligned}$$

Thus, for every $x \in B(0, 1)$,

$$\|Tx\| = \frac{1}{r} \left\| T \underbrace{(rx)}_{\in B(0, r)} \right\| \leq \frac{1}{r} (K + M_{x_0}) =: M,$$

so its clear $\|T\| \leq M$ for every $T \in \mathcal{F}$. ■

↪ **Theorem 2.7** (Banach-Saks-Steinhaus): Let X a Banach space and Y a nvs. Let $\{T_n\} \subseteq \mathcal{L}(X, Y)$. Suppose for every $x \in X$, $\lim_{n \rightarrow \infty} T_n(x)$ exists in Y . Then,

- $\{T_n\}$ are uniformly bounded in $\mathcal{L}(X, Y)$;
- For $T : X \rightarrow Y$ defined by $T(x) := \lim_{n \rightarrow \infty} T_n(x)$, we have $T \in \mathcal{L}(X, Y)$;
- $\|T\| \leq \liminf_{n \rightarrow \infty} \|T_n\|$ (*lower semicontinuity result*).

PROOF. (a) For every $x \in X$, $T_n(x) \rightarrow T(x)$ so $\|Tx\| < \infty$ hence $\sup_n \|T_n x\| < \infty$. By uniform boundedness, then, we find $\sup_n \|T_n\| =: C < \infty$.

(b) T is linear (by linearity of T_n). By (a),

$$\|T_n x\| \leq C \|x\|,$$

for every n, x , so

$$\|Tx\| \leq C \|x\| \quad \forall x \in X,$$

so T bounded.

(c) We know

$$\|T_n x\| \leq \|T_n\| \|x\| \quad \forall x \in X,$$

so

$$\frac{\|T_n x\|}{\|x\|} \leq \|T_n\|,$$

so

$$\liminf_n \frac{\|T_n x\|}{\|x\|} = \frac{\|Tx\|}{\|x\|} \leq \liminf_n \|T_n\|,$$

so by “suping” both sides,

$$\|T\| \leq \liminf_n \|T_n\|.$$

■

Remark 2.3:

- We do not necessarily have $T_n \rightarrow T$ in $\mathcal{L}(X, Y)$ i.e. with respect to the operator norm.
- If Y is a Banach space, then $\lim_{n \rightarrow \infty} T_n(x)$ exists in $Y \Leftrightarrow \{T_n x\}$ Cauchy in Y for every $x \in X$.

§2.5 Introduction to Hilbert Spaces

↪ **Definition 2.6** (Inner Product): An *inner product* on a vector space X is a map $(\cdot, \cdot) : X \times X \rightarrow \mathbb{R}$ such that for every $\lambda, \mu \in \mathbb{R}$ and $x, y, z \in X$,

- $(\lambda x + \mu y, z) = \lambda(x, z) + \mu(y, z)$;
- $(x, y) = (y, x)$;
- $(x, x) \geq 0$ and $(x, x) = 0 \Leftrightarrow x = 0$.

Remark 2.4: The first and second conditions combined imply that (\cdot, \cdot) actually *bilinear*, namely, linear in both coordinates.

Remark 2.5: An inner product induces a norm on a vector space by

$$\|x\| := (x, x)^{\frac{1}{2}}.$$

↪ **Proposition 2.2** (Cauchy-Schwarz Inequality): Any inner product satisfies Cauchy-Schwarz, namely,

$$|(x, y)| \leq \|x\| \|y\|,$$

for every $x, y \in X$.

PROOF. Suppose first $y = 0$. Then, the right hand side is clearly 0, and by linearity $(x, y) = 0$, hence we have $0 \leq 0$ and are done. Suppose then $y \neq 0$. Then, let $z = x - \frac{(x, y)}{(y, y)}y$ where $y \neq 0$. Then,

$$\begin{aligned} 0 \leq \|z\|^2 &= \left(x - \frac{(x, y)}{(y, y)}y, x - \frac{(x, y)}{(y, y)}y \right) \\ &= (x, x) - \frac{(x, y)}{(y, y)}(x, y) - \frac{(x, y)}{(y, y)}(y, x) + \frac{(x, y)^2}{(y, y)^2}(y, y) \\ &= (x, x) - \frac{2((x, y))^2}{(y, y)} + \frac{(x, y)^2}{(y, y)} \\ &= \|x\|^2 - \frac{(x, y)^2}{(y, y)} \\ &\Rightarrow \frac{(x, y)^2}{(y, y)} \leq \|x\|^2 \Rightarrow (x, y)^2 \leq \|x\|^2 \|y\|^2 \\ &\Rightarrow |(x, y)| \leq \|x\| \|y\|. \end{aligned}$$

■

↪ **Corollary 2.4:** The function $\|x\| := (x, x)^{\frac{1}{2}}$ is actually a norm on X .

PROOF. By definition, $\|x\| \geq 0$ and equal to zero only when $x = 0$. Also,

$$\|\lambda x\| = (\lambda x, \lambda x)^{\frac{1}{2}} = |\lambda|(x, x)^{\frac{1}{2}} = |\lambda|\|x\|.$$

Finally,

$$\begin{aligned} \|x + y\|^2 &= (x + y, x + y) \\ &= (x, x) + 2(x, y) + (y, y) \\ &= \|x\|^2 + \|y\|^2 + 2(x, y) \\ \text{by Cauchy-Schwarz} \quad &\leq \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \\ &= (\|x\| + \|y\|)^2, \end{aligned}$$

hence by taking square roots we see $\|x + y\| \leq \|x\| + \|y\|$ as desired. ■

↪ **Proposition 2.3** (Parallelogram Law): Any inner product space satisfies the following:

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

↪ **Corollary 2.5:** (\cdot, \cdot) is continuous, i.e. if $x_n \rightarrow x$ and $y_n \rightarrow y$, then $(x_n, y_n) \rightarrow (x, y)$.

PROOF.

$$\begin{aligned} |(x_n, y_n) - (x, y)| &= |(x_n, y_n) - (x, y_n) + (x, y_n) - (x, y)| \\ &= |(x_n - x, y_n) + (x, y_n - y)| \\ &\leq |(x_n - x, y_n)| + |(x, y_n - y)| \\ \text{(Cauchy-Schwarz)} \quad &\leq \underbrace{\|x_n - x\|}_{\rightarrow 0} \underbrace{\|y_n\|}_{\leq M} + \|x\| \underbrace{\|y_n - y\|}_{\rightarrow 0} \rightarrow 0. \end{aligned}$$
■

↪ **Definition 2.7** (Hilbert Space): A *Hilbert Space* H is a complete inner product space, namely, it is complete with respect to the norm induced by the inner product.

⊗ **Example 2.1:**

1. ℓ^2 , the space of square-summable real-valued sequences, equipped with inner product $(x, y) = \sum_{i=1}^{\infty} x_i y_i$.
2. L^2 , with inner product $(f, g) = \int f(x)g(x) dx$.

↪ **Definition 2.8** (Orthogonality): We say x, y *orthogonal* and write $x \perp y$ if $(x, y) = 0$. If $M \subseteq H$, then the *orthogonal complement* of M , denoted M^\perp , is the set

$$M^\perp = \{y \in H \mid (x, y) = 0, \forall x \in M\}.$$

Remark 2.6: M^\perp is always a closed subspace of H . If $y_1, y_2 \in M^\perp$, then for every $x \in M$,

$$(x, \alpha y_1 + \beta y_2) = \alpha(x, y_1) + \beta(x, y_2) = 0,$$

so M^\perp a subspace.

If $y_n \rightarrow y$ in the norm on H and $\{y_n\} \subseteq M^\perp$, then using the continuity of (\cdot, \cdot) , we know that for every $x \in M$, $(x, y_n) \rightarrow (x, y)$. But then $(x, y_n) = 0$ for every n and thus $(x, y) = 0$ so $y \in M^\perp$, hence M^\perp closed.

↪ **Proposition 2.4:** If $M \subsetneq H$ is a closed subspace, then every $x \in H$ has a unique decomposition

$$x = u + v, \quad u \in M, v \in M^\perp.$$

Hence, we may write $H = M \oplus M^\perp$. Moreover,

$$\|x - u\| = \inf_{y \in M} \|x - y\|, \quad \|x - v\| = \inf_{y \in M^\perp} \|x - y\|.$$

PROOF. Let $x \in H$. If $x \in M$, we're done with $u = x, v = 0$. Else, if $x \notin M$, then we claim that there is some $u \in M$ such that $\|x - u\| = \inf_{y \in M} \|x - y\| =: \delta > 0$. By definition of the infimum, there exists a sequence $\{u_n\} \subseteq M$ such that

$$\|x - u_n\|^2 \leq \delta^2 + \frac{1}{n}.$$

Let $\bar{x} := u_m - x, \bar{y} = u_n - x$. By the Parallelogram Law,

$$\|\bar{x} - \bar{y}\|^2 + \|\bar{x} + \bar{y}\|^2 = 2\|\bar{x}\|^2 + 2\|\bar{y}\|^2$$

hence

$$\|u_m - u_n\|^2 + \|u_m + u_n - 2x\|^2 = 2\|u_m - x\|^2 + 2\|u_n - x\|^2.$$

Now, the second term can be written

$$\|u_m + u_n - 2x\|^2 = 4 \left\| \frac{u_m + u_n}{2} - x \right\|^2,$$

hence we find

$$\|u_m - u_n\|^2 = 2\|u_m - x\|^2 + 2\|u_n - x\|^2 - 4 \left\| \frac{u_m + u_n}{2} - x \right\|^2.$$

Recall that M a subspace, hence $\frac{1}{2}(u_m + u_n) \in M$ so $\|x - \frac{1}{2}(u_m + u_n)\| \geq \delta$ as defined before. Thus, we find that by our choice of $\{u_n\}$,

$$\|u_m - u_n\|^2 \leq 2\left(\delta^2 + \frac{1}{m}\right) + 2\left(\delta^2 + \frac{1}{n}\right) - 4\delta^2 = \frac{2}{m} + \frac{2}{n},$$

and thus, by making m, n sufficiently large we can make $\|u_m - u_n\|$ arbitrarily small. Hence, $\{u_n\} \subseteq M$ are Cauchy. H is complete, hence the $\{u_n\}$'s converge, and thus since M closed, $u_n \rightarrow u \in M$. Then, we find

$$\begin{aligned} \|x - u\| &\leq \|x - u_n\| + \|u_n - u\| \\ &\leq \underbrace{\left(\delta^2 + \frac{1}{n}\right)^{\frac{1}{2}}}_{\rightarrow \delta} + \underbrace{\|u_n - u\|}_{\rightarrow 0} \rightarrow \delta. \end{aligned}$$

But also, $u \in M$ and thus $\|x - y\| \geq \delta$, and we conclude $\|x - u\| = \delta = \inf_{y \in M} \|x - y\|$.

Next, we claim that if we define $v = x - y$, then $v \in M^\perp$. Consider $y \in M, t \in \mathbb{R}$, then

$$\left\|x - \underbrace{(u - ty)}_{\in M}\right\|^2 = \|v + ty\|^2 = \|v\|^2 + 2t(v, y) + t^2\|y\|^2.$$

Then, notice that the map

$$t \mapsto \|v + ty\|^2$$

is minimized when $t = 0$, since $\|x - z\|$ for $z \in M$ is minimized when $z = u$, as we showed in the previous part, so equivalently $\|x - (u - ty)\|^2$ minimized when $t = 0$. Thus,

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \|v + ty\|^2|_{t=0} = \frac{\partial}{\partial t} [\|v\|^2 + 2t(v, y) + t^2\|y\|^2]|_{t=0} \\ &= (2(v, y) + 2t\|y\|^2)|_{t=0} = (v, y) \\ &\Rightarrow (v, y) = 0 \forall y \in M \Rightarrow v \in M^\perp. \end{aligned}$$

So, $x = u + v$ and $u \in M, v \in M^\perp$. For uniqueness, suppose $x = u_1 + v_1 = u_2 + v_2$. Then, $u_1 - u_2 = v_2 - v_1$, but then

$$\|v_2 - v_1\|^2 = (v_2 - v_1, v_2 - v_1) = (v_2 - v_1, u_2 - u_1) = 0,$$

so $v_2 = v_1$ so it follows $u_2 = u_1$ and uniqueness holds. ■

↪ **Definition 2.9** (Dual of H): The *dual* of H , denoted H^* , is the set

$$H^* := \{f : H \rightarrow \mathbb{R} \mid f \text{ continuous and linear}\}.$$

On this space, we may equip the operator norm

$$\|f\|_{H^*} = \|f\| = \sup_{x \in H} \frac{|f(x)|}{\|x\|_H} = \sup_{\|x\| \leq 1} |f(x)|.$$

⊗ **Example 2.2:** For $y \in H$, let $f_y : H \rightarrow \mathbb{R}$ be given by $f_y(x) = (x, y)$. By CS,

$$\|f_y\|_{H^*} = \sup_{\|x\| \leq 1} (x, y) \leq \sup_{\|x\| \leq 1} \|x\| \|y\| \leq \|y\|.$$

Also, if $y \neq 0$, then

$$f_y\left(\frac{y}{\|y\|}\right) = \left(\frac{y}{\|y\|}, y\right) = \|y\|.$$

Thus, $\|f_y\|_{H^*} = \|y\|_H$. It turns out all such functionals are of this form.

↪ **Theorem 2.8** (Riesz Representation for Hilbert Spaces): If $f \in H^*$, there exists a unique $y \in H$ such that $f(x) = (x, y)$ for every $x \in X$.

PROOF. We show first existence. If $f \equiv 0$, then $y = 0$. Otherwise, let $M = \{x \in X \mid f(x) = 0\} = f^{-1}(\{0\})$, so $M \subsetneq H$. f linear, so M a linear subspace. f is continuous, so in addition M is closed. By the previous theorem, $M^\perp \neq \{0\}$. Let $z \in M^\perp$ of norm 1.

Fix $x \in H$, and define

$$u := f(x)z - f(z)x.$$

Then, notice that by linearity

$$f(u) = f(x)f(z) - f(z)f(x) = 0,$$

so $u \in M$. Thus, since $z \in M^\perp$, $(u, z) = 0$, so in particular,

$$\begin{aligned} (u, z) = 0 &= (f(x)z - f(z)x, z) \\ &= f(x)(z, z) - f(z)(x, z) \\ &= f(x)\|z\|^2 - (x, f(z)z) \\ &= f(x) - (x, f(z)z), \end{aligned}$$

hence, rearranging we find

$$f(x) = (x, f(z)z),$$

and thus letting $y = f(z)z$ completes the proof of existence, noting z independent of x .

For uniqueness, suppose $(x, y) = (x, y')$ for every $x \in X$. Then, $(x, y - y') = 0$ for every $x \in X$, hence letting $x = y - y'$ we conclude $(y - y', y - y') = 0$ thus $y - y' = 0$ so $y = y'$, and uniqueness holds. ■

↪ **Definition 2.10** (Orthonormal Set): A collection $\{e_j\} \subseteq H$ is *orthonormal* if $(e_i, e_j) = \delta_i^j$.

Remark 2.7: The following section writes notations assuming H has a countable basis. However, for more general Hilbert spaces, all countable summations can be replaced with uncountable ones in which only countably many elements are nonzero. The theory is very similar.

↪ **Definition 2.11** (Orthonormal Basis): A collection $\{e_j\} \subseteq H$ is an *orthonormal basis* for H if $\{e_j\}$ is an orthonormal set, and $x = \sum_{j=1}^{\infty} (x, e_j) e_j$ for every $x \in H$, in the sense that

$$\left\| x - \sum_{j=1}^N (x, e_j) e_j \right\| \rightarrow 0, \quad N \rightarrow \infty.$$

↪ **Theorem 2.9** (General Pythagorean Theorem): If $\{e_j\}_{j=1}^{\infty} \subseteq H$ are orthonormal and $\{\alpha_i\}_{i=1}^{\infty} \subseteq \mathbb{R}$ are orthonormal, then for any N ,

$$\left\| \sum_{i=1}^N \alpha_i e_i \right\|^2 = \sum_{i=1}^N |\alpha_i|^2.$$

PROOF.

$$\left\| \sum_{i=1}^N \alpha_i e_i \right\|^2 = \left(\sum_{i=1}^N \alpha_i e_i, \sum_{j=1}^N \alpha_j e_j \right) = \sum_{i=1}^N \sum_{j=1}^N \alpha_i \alpha_j \underbrace{(e_i, e_j)}_{=\delta_i^j} = \sum_{i=1}^N \alpha_i^2.$$

■

We can also **Gram-Schmidt** in infinite-dimensional Hilbert spaces. Let $\{x_i\} \subseteq H$. Let

$$e_1 = \frac{x_1}{\|x_1\|},$$

and inductively, for any $n \geq 2$, define

$$v_N = x_N - \sum_{i=1}^{N-1} (x_N, e_i) e_i.$$

Then, for any N , $\text{span}(v_1, \dots, v_N) = \text{span}(e_1, \dots, e_N)$, and for any $j < N$,

$$(v_N, e_j) = (x_N, e_j) - \sum_{i=1}^N (x_N, e_i) (e_i, e_j) = (x_N, e_j) - (x_N, e_j) = 0.$$

Let then $e_N = \frac{v_N}{\|v_N\|}$. Then, $\{e_i\}_{i=1}^\infty$ will be orthonormal; we discuss how to establish when this set will actually be a basis to follow.

→ **Theorem 2.10** (Bessel's Inequality): If $\{e_i\}_{i=1}^\infty$ are orthonormal, then for any $x \in H$,

$$\sum_{i=1}^{\infty} |(x, e_i)|^2 \leq \|x\|^2.$$

PROOF. We have

$$\begin{aligned} 0 &\leq \left\| x - \sum_{i=1}^N (x, e_i) e_i \right\|^2 \\ &= \left(x - \sum_{i=1}^N (x, e_i) e_i, x - \sum_{j=1}^N (x, e_j) e_j \right) \\ &= \|x\|^2 - 2 \sum_{i=1}^N (x, e_i)^2 + \sum_{i=1}^N (x, e_i)^2 \\ &= \|x\|^2 - \sum_{i=1}^N (x, e_i)^2, \end{aligned}$$

so $\sum_{i=1}^N (x, e_i)^2 \leq \|x\|^2$; letting $N \rightarrow \infty$ proves the desired inequality, since the RHS is independent of N . ■

→ **Theorem 2.11**: If $\{e_i\}_{i=1}^\infty$ are orthonormal, then TFAE:

- (a) completeness: if $(x, e_i) = 0$ for every i , then $x = 0$, the zero vector;
- (b) Parseval's identity holds: $\|x\|^2 = \sum_{i=1}^\infty (x, e_i)^2$ for every $x \in H$;
- (c) $\{e_i\}_{i=1}^\infty$ form a basis for H , i.e. $x = \sum_{i=1}^\infty (x, e_i) e_i$ for every $x \in H$.

PROOF. ((a) \Rightarrow (c)) By Bessel's, $\sum_{i=1}^\infty (x, e_i)^2 \leq \|x\|^2$. So, for any $M \geq N$,

$$\left\| \sum_{i=N}^M (x, e_i) e_i \right\|^2 = \sum_{i=N}^M (x, e_i)^2,$$

which must converge to zero as $N, M \rightarrow \infty$, since the whole series converges (being bounded). Hence, $\left\{ \sum_{i=1}^N (x, e_i) e_i \right\}_N$ is Cauchy in $\|\cdot\|$ and since H complete, $\sum_{i=1}^\infty (x, e_i) e_i$ converges in H . Putting $y = x - \sum_{i=1}^\infty (x, e_i) e_i$, we find

$$(y, e_i) = (x, e_i) - (x, e_i) = 0 \quad \forall i,$$

hence by assumption in (a), it follows that $y = 0$ so $x = \sum_{i=1}^\infty (x, e_i) e_i$ and thus $\{e_i\}$ a basis for H and (c) holds.

((c) \Rightarrow (b)) Since $x = \sum_{i=1}^\infty (x, e_i) e_i$, then,

$$\|x\|^2 - \sum_{i=1}^N (x, e_i)^2 = \left\| x - \sum_{i=1}^N (x, e_i) e_i \right\|^2 \rightarrow 0$$

as $N \rightarrow \infty$, hence $\|x\|^2 = \sum_{i=1}^{\infty} (x, e_i)^2$.

((b) \Rightarrow (a)) If $(x, e_i) = 0$ for every i , then by Parseval's $\|x\|^2 = \sum_{i=1}^{\infty} 0 = 0$ so $x = 0$. ■

Remark 2.8: (a) is equivalent to $\text{span}(e_1, e_2, \dots)$ is *dense* in H .

↪ **Theorem 2.12:** Every Hilbert space has an orthonormal basis.

PROOF. Let $\mathcal{F} = \{\text{orthonormal subsets of } H\}$. \mathcal{F} can be (partially) ordered by inclusion, as can be upper bounded by the union over the whole space. By Zorn's Lemma, there is a maximal set in \mathcal{F} , which implies completeness, (a). ■

↪ **Proposition 2.5:** H is separable iff H has a countable basis.

PROOF. (\Leftarrow) If H has a countable basis $\{e_j\}$, $\text{span}_{\mathbb{Q}}\{e_j\}$ is a countable dense set.

(\Rightarrow) If H is separable, let $\{x_n\}$ be a countable dense set. Use Gram-Schmidt, to produce a countable, orthonormal set, which is dense and hence a (countable) basis for H . ■

Remark 2.9: All this can be extended to uncountable bases.

§2.6 Adjoints, Duals and Weak Convergence (for Hilbert Spaces)

First consider $T : H \rightarrow H$ bounded and linear. Fix $y \in H$. We claim that the map

$$x \mapsto (T(x), y)$$

belongs to H^* , namely is bounded and linear. Linearity is clear since T linear. We know by Cauchy-Schwarz that

$$|(T(x), y)| \leq \|T(x)\| \|y\| \leq \|T\| \|x\| \|y\| \leq C \|x\|,$$

so indeed $x \mapsto (T(x), y) \in H^*$. By Riesz Representation Theorem, there is some unique $z \in H$ such that

$$(T(x), y) = (x, z) \quad \forall x \in H.$$

This motivates the following.

↪ **Definition 2.12** (Adjoint of T): Let $T^* : H \rightarrow H$ be defined by

$$(Tx, y) = (x, T^*y), \quad \forall x, y \in H.$$

Remark 2.10: In finite dimensions, T can be identified with some $n \times n$ matrix, in which case $T^* = T^t$, the transpose of T ; namely $Tx \cdot b = x \cdot T^t b$.

↪ **Proposition 2.6:** If $T \in \mathcal{L}(H) := \mathcal{L}(H, H)$, then $T^* \in \mathcal{L}(H)$ and $\|T^*\| = \|T\|$.

PROOF. Linearity of T^* is clear. Also, for any $\|y\| \leq 1$,

$$\|T^*y\|^2 = (T^*y, T^*y) = (TT^*y, y) \leq \|T\|\|T^*(y)\|\|y\|$$

so $\|T^*y\| \leq \|T\|$ for all $\|y\| = 1$. so $\|T^*\| \leq \|T\|$ hence $T^* \in \mathcal{L}(H)$. But also, if $x \in H$ with $\|x\| = 1$, then symmetrically,

$$\|Tx\|^2 = (Tx, Tx) = (x, T^*Tx) \leq \|T^*\|\|Tx\|$$

so similarly $\|T\| \leq \|T^*\|$ hence equality holds. ■

↪ **Proposition 2.7:** $(T^*)^* = T$.

PROOF. On the one hand,

$$(T^*y, x) = (y, (T^*)^*x) = ((T^*)^*x, y)$$

while also

$$(T^*y, x) = (x, T^*y) = (Tx, y)$$

so $(Tx, y) = ((T^*)^*x, y)$, from which it follows that $T = T^{**}$. ■

↪ **Proposition 2.8:** $(T + S)^* = T^* + S^*$, and $(T \circ S)^* = S^* \circ T^*$.

We'll write $N(T)$ for the nullspace/kernel of T , and $R(T)$ for the range/image of T .

↪ **Proposition 2.9:** Suppose $T \in \mathcal{L}(H)$. Then,

- $N(T^*) = R(T)^\perp$ (and hence, if $R(T)$ closed, $H = N(T^*) \oplus R(T)$);
- $N(T) = R(T^*)^\perp$ (and hence, if $R(T^*)$ closed, $H = N(T) \oplus R(T^*)$).

PROOF. $N(T^*) = \{y \in H : T^*y = 0\}$, so if $y \in N(T^*)$, $(Tx, y) = (x, T^*y) = (x, 0) = 0$, which holds iff y orthogonal to Tx , and since this holds for all $x \in H$, $y \in R(T)^\perp$.

Then, if $R(T)$ closed, the by orthogonal decomposition we'll find $H = R(T) \oplus R(T)^\perp = R(T) \oplus N(T^*)$.

The other claim follows similarly. ■

Remark 2.11: Recall that $R(T)^\perp$ is closed; hence

$$(R(T)^\perp)^\perp = \{z \in H \mid (y, z) = 0 \forall y \in R(T)^\perp\},$$

and is also closed; hence $(R(T)^\perp)^\perp = \overline{R(T)}$ thus equivalently $N(T^*)^\perp = \overline{R(T)}$.

Remark 2.12: By the Closed Graph Theorem, T linear and bounded gives T closed; namely, the graph of T closed; this is *not* the same as saying the range of T closed.

⊗ **Example 2.3:** Consider $C([0, 1]) \subseteq L^2([0, 1])$, and $T : C([0, 1]) \rightarrow L^2([0, 1])$ given by the identity, $Tf = f$. Then, T is bounded, but $R(T) = C([0, 1])$; this subspace is *not* closed in $L^2([0, 1])$, since there exists sequences of continuous functions that converge to an L^2 , but not continuous, function.

Remark 2.13: The prior theorem is key in “solvability”, especially if T a differential or integral operator. If we wish to find u such that $Tu = f$, we need that $f \in R(T)$, hence $f \in N(T^*)^\perp$.

⊗ **Example 2.4:** Let $M \subsetneq H$ a closed linear subspace. Then, $H = M \oplus M^\perp$; define the projection operator

$$P : H \rightarrow H, \quad x = u + v \in M \oplus M^\perp \mapsto u.$$

This means, in particular, $x = Px + (\text{id} - P)x$. We claim $P \in \mathcal{L}(H)$, $\|P\| = 1$, $P^2 = P$, and $P^* = P$.

Linearity is clear. To show $P^2 = P$, write $x = Px + v$. Then, composing both sides with P , we find $Px = P^2x + Pv = P^2x$, so $Px = P^2x$ for every $x \in H$. To see the norm, we find that for every $x \in H$,

$$\begin{aligned} \|x\|^2 &= (x, x) = (Px + (\text{id} - P)x, Px + (\text{id} - P)x) \\ &= \|Px\|^2 + 2\underbrace{(Px, (\text{id} - P)x)}_{\perp} + \|(\text{id} - P)x\|^2 \\ &= \|Px\|^2 + \|(\text{id} - P)x\|^2 \geq \|Px\|^2 \\ &\Rightarrow \|Px\| \leq \|x\| \Rightarrow \|P\| \leq 1, \end{aligned}$$

and moreover if $x \in M$, $Px = x$ so $\|Px\| = \|x\|$ hence $\|P\| = 1$ indeed.

Finally, to show P self-adjoint, let $x, y \in H$, then,

$$0 = (Px, (\text{id} - P)y) = (Px, y - Py) \Rightarrow (Px, y) = (Px, Py).$$

Symmetrically, $(x, Py) = (Px, Py)$, hence $(Px, y) = (x, Py)$, and so $P = P^*$.

§2.7 Introduction to Weak Convergence

We let throughout X be a Banach space.

↪ **Definition 2.13** (Weak convergence): We say $\{x_n\} \subseteq X$ converges weakly to $x \in X$, and write

$$x_n \rightharpoonup x$$

iff for every $f \in X^* = \{f : X \rightarrow \mathbb{R} \text{ bounded, linear}\}$, $f(x_n) \rightarrow f(x)$.

↪ **Definition 2.14** (Weak topology $\sigma(X, X^*)$): The weak topology $\sigma(X, X^*)$ is the weak topology induced by

$$\mathcal{F} = X^*.$$

In particular, this is the smallest topology in which every f continuous.

Recall that this was defined as being $\tau(\{f^{-1}(\mathcal{O})\})$ for \mathcal{O} open in \mathbb{R} . A base for this topology is given by $\mathcal{B} = \{\text{finite intersections of } \{f^{-1}(\mathcal{O})\}\}$. Namely, let $\mathcal{B}_X := \{B_{\varepsilon, f_1, f_2, \dots, f_n}(x)\}$ where

$$B_{\varepsilon, f_1, f_2, \dots, f_n}(x) = \{x' \in X \mid |f_k(x') - f_k(x)| < \varepsilon, \forall 1 \leq k \leq n\}.$$

So, $x_n \rightarrow x$ in $\sigma(X, X^*)$ if for every $\varepsilon > 0$, and ball $B_{\varepsilon, f_1, \dots, f_m}(x)$, there is an N such that for every $n \geq N$, $x_n \in B_{\varepsilon, f_1, \dots, f_m}(x)$, hence for every $f \in X^*$, $|f(x_n) - f(x)| < \varepsilon$.

For Hilbert spaces, by Riesz we know $f \in H^*$ can always be identified with $f(x) = (x, y)$ for some $y \in H$. So, we find $x_n \rightharpoonup x$ in H iff for every $y \in H$, $(x_n, y) \rightarrow (x, y)$.

Remark 2.14: If $x_n \rightarrow x$ in H , then $(x_n, y) \rightarrow (x, y)$; so this “normal” (we say “strong”) convergence implies weak convergence.

↪ **Proposition 2.10:** (i) Suppose $x_n \rightharpoonup x$ in H . Then, $\{x_n\}$ are bounded in H , and $\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$.

(ii) If $y_n \rightarrow y$ (strongly) in H and $x_n \rightharpoonup x$ (weakly) in H , then $(x_n, y_n) \rightarrow (x, y)$.

Remark 2.15: It does *not* hold, though, that $x_n \rightharpoonup x$, $y_n \rightharpoonup y$ gives $(x_n, y_n) \rightarrow (x, y)$.

PROOF. (i) If $x_n \rightharpoonup x$, then

$$\left(x_n, \frac{x}{\|x\|}\right) \rightarrow \left(x, \frac{x}{\|x\|}\right) = \|x\|.$$

By Cauchy-Schwarz, we also have

$$\left|\left(x_n, \frac{x}{\|x\|}\right)\right| \leq \|x_n\| \left(\frac{\|x\|}{\|x\|}\right) = \|x_n\|,$$

hence we conclude

$$\liminf_{n \rightarrow \infty} \left(x_n, \frac{x}{\|x\|} \right) \leq \liminf_{n \rightarrow \infty} \|x_n\| \Rightarrow \|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|.$$

To argue $\{x_n\}$ bounded, we need the uniform boundedness principle. We can view $\{x_n\} \subseteq H^{**}$ by the canonical association $x_n^{**} : f \mapsto f(x_n)$. Since $f \in H^*$, there is a y such that $f(\cdot) = (\cdot, y)$; label $f = f_y$. Then, for every $f \in H^*$,

$$x_n^{**}(f_y) = f_y(x_n) = (x_n, y) \rightarrow (x, y),$$

by weak convergence. Hence, it must be that $\sup_n |x_n^{**} f| = \sup_n |f_y(x_n)| < \infty$ for every $f \in H^*$, namely $\{x_n^{**}\}$ is a pointwise-bounded family of functions. Thus, by uniform boundedness, there is a $C > 0$ such that $|x_n^{**} f| \leq C \|f\|$ for every $f \in H^*$ and $n \geq 1$. In particular, if we take $f(\cdot) := (\cdot, x_n)$, we know by Riesz that $\|f\| = \|x_n\|$ on the one hand, so for every $n \geq 1$,

$$C\|f\| = C\|x_n\| \geq |x_n^{**} f| = |(x_n, x_n)| = \|x_n\|^2 \Rightarrow \|x_n\| \leq C,$$

completing the claim of boundedness.

(ii) If $y_n \rightarrow y$ in H ,

$$\begin{aligned} |(x_n, y_n) - (x, y)| &\leq |(x_n, y_n - y)| + |(x_n - x, y)| \\ &\leq \underbrace{\|x_n\|}_{\text{bounded}} \underbrace{\|y_n - y\|}_{\rightarrow 0} + \underbrace{|(x_n - x, y)|}_{\rightarrow 0 \text{ by weak}} \rightarrow 0. \end{aligned}$$

■

The real help of weak convergence is in the ease of achieving weak compactness;

↪ **Theorem 2.13** (Weak Compactness): Every bounded sequence in H has a weakly convergent subsequence.

↪ **Theorem 2.14** (Helley's Theorem): Let X a separable normed vector space and $\{f_n\} \subseteq X^*$ such that there is a constant $C > 0$ such that $|f_n(x)| \leq C\|x\|$ for every $x \in X$ and $n \geq 1$. Then, there exists a subsequence $\{f_{n_k}\}$ and an $f \in X^*$ such that $f_{n_k}(x) \rightarrow f(x)$ for every $x \in X$.

PROOF. This is essentially a specialization of the Arzelà-Ascoli lemma. To apply it, we need X separable (done), the sequence to be pointwise bounded (done), and the sequence to be equicontinuous. To verify this last one, we know that

$$\|f_n(x)\| \leq C\|x\| \Rightarrow \|f_n\| \leq C, \forall n \geq 1,$$

hence by linearity, for any $x, y \in X$,

$$\|f_n(x) - f_n(y)\| \leq C\|x - y\|, \forall n \geq 1,$$

so in particular $\{f_n\}$ uniformly Lipschitz, thus equicontinuous.

■

PROOF. (Of [Thm. 2.13](#)) Let $\{x_n\} \subseteq H$ be bounded and let $H_0 = \overline{\text{span}\{x_1, \dots, x_n, \dots\}}$, so H_0 is separable, and $(H_0, (\cdot, \cdot))$ is a Hilbert space (being closed). Let $f_n \in H_0^*$ be given by

$$f_n(x) = (x_n, x), \forall x \in H_0.$$

Then,

$$|f_n(x)| \leq \|x_n\| \|x\| \leq C \|x\|,$$

since $\{x_n\}$ bounded by assumption. By Helly's Theorem, then, there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k}(x) \rightarrow f(x)$ for every $x \in H_0$, where $f \in H_0^*$. By Riesz, then, $f(x) = (x, x_0)$ for some $x_0 \in H_0$. This implies

$$(x_{n_k}, x) \rightarrow (x_0, x), \forall x \in H_0.$$

Let P the projection of H onto H_0 . Then, for every $x \in H$,

$$(x_{n_k}, (\text{id} - P)x) = (x_0, (\text{id} - P)x) = 0$$

so for any $x \in H$,

$$\begin{aligned} \lim_{k \rightarrow \infty} (x_{n_k}, x) &= \lim_{k \rightarrow \infty} (x_{n_k}, Px + (\text{id} - P)x) \\ &= \lim_{k \rightarrow \infty} (x_{n_k}, \underbrace{Px}_{\in H_0}) \\ &= (x_0, Px) = (x_0, Px + (\text{id} - P)x) = (x_0, x), \end{aligned}$$

as we aimed to show. ■

§2.8 Review of L^p Spaces

We always consider $\Omega \subseteq \mathbb{R}^d$.

↪ **Definition 2.15** ($L^p(\Omega)$): For $1 \leq p < \infty$, define

$$L^p(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{R} \mid f \text{ measurable and } \int_{\Omega} |f|^p dx < \infty \right\},$$

endowed with the norm

$$\|f\|_{L^p(\Omega)} = \|f\|_p := \left[\int_{\Omega} |f(x)|^p dx \right]^{\frac{1}{p}}.$$

For $p = \infty$, define

$$L^\infty(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ measurable and } \exists C < \infty \text{ s.t. } |f| \leq C \text{ a.e.}\},$$

endowed with the norm

$$\|f\|_{L^\infty(\Omega)} = \|f\|_\infty := \inf\{C : |f| \leq C \text{ a.e.}\}.$$

The following are recalled but not proven here, see [here](#).

↪ **Theorem 2.15** (Holder's Inequality): For $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, then if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, then $fg \in L^1(\Omega)$, and

$$\int |fg| \, dx \leq \|f\|_p \|g\|_q.$$

↪ **Theorem 2.16** (Minkowski's Inequality): For all $1 \leq p \leq \infty$, $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. In particular, $L^p(\Omega)$ is a normed vector space.

↪ **Theorem 2.17** (Riesz-Fischer Theorem): $L^p(\Omega)$ is a Banach space for every $1 \leq p \leq \infty$.

↪ **Theorem 2.18**: $C_c(\mathbb{R}^d)$, the space of continuous functions with compact support, simple functions, and step functions are all dense subsets of $L^p(\mathbb{R}^d)$, for every $1 \leq p < \infty$.

↪ **Theorem 2.19** (Separability of $L^p(\Omega)$): L^p is separable, for every $1 \leq p < \infty$.

PROOF. We prove for $\Omega = \mathbb{R}^d$. Let

$$\mathcal{R} := \left\{ \prod_{i=1}^d (a_i, b_i) \mid a_i, b_i \in \mathbb{Q} \right\},$$

and let

$$\mathcal{E} := \{\text{finite linear combinations of } \chi_R \text{ for } R \in \mathcal{R} \text{ with coefficients in } \mathbb{Q}\},$$

where χ_R the indicator function of the set R . Then, we claim \mathcal{E} dense in $L^p(\mathbb{R}^d)$.

Given $f \in L^p(\mathbb{R}^d)$ and $\varepsilon > 0$, by density of $C_c(\mathbb{R}^d)$ there is some f_1 with $\|f - f_1\|_p < \varepsilon$. Let $\text{supp}(f_1) \subseteq R \in \mathcal{R}$. Now, let $\delta > 0$. Write

$$R = \cup_{i=1}^N R_i, \quad R_i \in \mathcal{R},$$

such that

$$\text{osc}_{R_i}(f_1) := \sup_{R_i} f_1 - \inf_{R_i} f_1 < \delta.$$

Then, let

$$f_2(x) = \sum_{i=1}^N q_i \chi(R_i), \quad q_i \in \mathbb{Q} \text{ s.t. } q_i \approx f_1|_{R_i},$$

so

$$\|f_2 - f_1\|_\infty < \delta.$$

Hence,

$$\begin{aligned}\|f_2 - f_1\|_p &\leq \left(\int_R |f_2(x) - f_1(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq \|f_1 - f_2\|_\infty \cdot m(R)^{\frac{1}{p}} < \delta \cdot m(R)^{\frac{1}{p}},\end{aligned}$$

where m the Lebesgue measure on \mathbb{R}^d . δ was arbitrary so we may take it arbitrarily small such that $\delta m(R)^{\frac{1}{p}} < \varepsilon$, hence for such a δ ,

$$\|f - f_2\|_p \leq \|f - f_1\|_p + \|f_1 - f_2\|_p < 2\varepsilon.$$

Now, $f_2 \in \mathcal{E}$, and thus \mathcal{E} is dense in $L^p(\mathbb{R}^d)$, and countable by construction, thus $L^p(\mathbb{R}^d)$ separable. ■

Remark 2.16: $L^\infty(\Omega)$ is *not* separable, and $C_c(\mathbb{R}^d)$ is *not* dense in $L^\infty(\Omega)$.

Remark 2.17 (Special Cases):

- If Ω has finite measure, $L^p(\Omega) \subseteq L^{p'}(\Omega)$ for every $p \geq p'$.
- $\ell^p := \left\{ a = (a_n)_{n=1}^\infty \mid \sum_{n=1}^\infty |a_n|^p < \infty \right\}$ endowed with the norm $|a|_{\ell^p} := \left(\sum_{n=1}^\infty |a_n|^p \right)^{1/p}$.

§2.9 $(L^p)^*$: The Riesz Representation Theorem

We are interested in functions $T : L^p(\Omega) \rightarrow \mathbb{R}$ which is bounded and linear. For instance, let $g \in L^q(\Omega)$ and $f \in L^p(\Omega)$ where p, q conjugates, and define

$$T(f) := \int_\Omega f(x)g(x) dx.$$

This is clearly linear, and by Holders,

$$|Tf| = \left| \int_\Omega fg \right| \leq \|f\|_p \|g\|_q.$$

so

$$\left| T \left(\frac{f}{\|f\|_p} \right) \right| \leq \|g\|_q, \quad \forall f \in L^p(\Omega), \Rightarrow \|T\| \leq \|g\|_q,$$

and thus $T \in (L^p(\Omega))^*$. Moreover, if $1 < p < \infty$, $1 < q < \infty$, let

$$f(x) = \frac{|g(x)|^{q-2} g(x)}{\|g\|_q^{q-1}}.$$

Then,

$$\begin{aligned}
\int_{\Omega} |f(x)|^p dx &= \frac{1}{\|g\|_q^{(q-1)p}} \int_{\Omega} |g(x)|^{(q-2)p} |g(x)|^p dx \\
&= \frac{1}{\|g\|_q^{(q-1)p}} \int_{\Omega} |g(x)|^{qp-p} dx.
\end{aligned}$$

Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $q + p = pq$, so further

$$= \frac{1}{\|g\|_q^q} \int_{\Omega} |g(x)|^q dx = \frac{1}{\|g\|_q^q} \cdot \|g\|_q^q = 1,$$

so f as defined indeed in $L^p(\Omega)$ and moreover has L^p -norm of 1. In addition,

$$\begin{aligned}
|Tf| &= \frac{1}{\|g\|_q^{q-1}} \int_{\Omega} |g(x)^{q-2}| g(x) g(x) dx \\
&= \frac{1}{\|g\|_q^{q-1}} \int_{\Omega} |g(x)|^q dx \\
&= \frac{1}{\|g\|_q^{q-1}} \|g\|_q^q = \|g\|_q,
\end{aligned}$$

so $\|T\| = \|g\|_q$ as desired. We have, more generally, akin to the Riesz representation theorem,

↪ **Theorem 2.20** (Riesz-Representation Theorem for $L^p(\Omega)$): Let $1 \leq p < \infty$. For any $T \in (L^p(\Omega))^*$, there exists a unique $g \in L^q(\Omega)$ such that $T(f) = \int_{\Omega} f(x)g(x) dx$ with $\|T\| = \|g\|_q$.

We'll only prove for $\Omega \subseteq \mathbb{R}$. First:

↪ **Proposition 2.11**: Let $T, S \in (L^p(\Omega))^*$. If $T = S$ on a dense subset $E \subseteq L^p(\Omega)$, then $T = S$ everywhere.

PROOF. Let $f_0 \in L^p(\Omega)$. By density, there exists $\{f_n\} \subseteq E$ such that $f_n \rightarrow f$ in $L^p(\Omega)$. By continuity, $Tf_n \rightarrow Tf_0$ and $Sf_n \rightarrow Sf_0$, while $Tf_n = Sf_n$ for every $n \geq 1$, so by uniqueness of limits in \mathbb{R} , $Tf_0 = Sf_0$. ■

The general outline of the proof of [Thm. 2.20](#) is the following:

- prove the theorem for f a step function;
- prove the theorem for f bounded and measurable;
- conclude the full theorem by appealing to the previous proposition.

To do this, we need first to recall the notion of *absolutely continuous functions*. Fix $[a, b] \subseteq \mathbb{R}$ and $G : [a, b] \rightarrow \mathbb{R}$. G is said to be absolutely continuous on $[a, b]$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every disjoint collection $\{(a_k, b_k)\}_{k=1}^N \subseteq [a, b]$ with $\sum_{k=1}^N (a_k - b_k) < \delta$, then $\sum_{k=1}^N |G(b_k) - G(a_k)| < \varepsilon$. In particular, we need the following result, proven [here](#):

↪ **Theorem 2.21:** If $G : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous, then $g = G'$ exists a.e. on $[a, b]$, $g \in L^1([a, b])$, and for every $x \in [a, b]$,

$$G(x) - G(a) = \int_a^x g(t) dt.$$

PROOF (Of [Thm. 2.20](#) with $\Omega = [a, b]$). Let $T \in (L^p([a, b]))^*$.

Step 1: Let f a step function. The function $\chi_{[a, x]} \in L^p([a, b])$; define

$$G_T(x) := T(\chi_{[a, x]}).$$

We claim G_T absolutely continuous. Consider $\{(a_k, b_k)\}_{k=1}^N$ disjoint. Then, for every $[c, d] \subseteq [a, b]$, $G_T(d) - G_T(c) = T(\chi_{[a, d]}) - T(\chi_{[a, c]}) = T(\chi_{[a, d]} - \chi_{[a, c]}) = T(\chi_{[c, d]})$, so

$$\begin{aligned} \sum_{k=1}^N (G_T(b_k) - G_T(a_k)) &= \sum_{k=1}^N c_k \cdot (G_T(b_k) - G_T(a_k)), \quad c_k := \operatorname{sgn}(G_T(b_k) - G_T(a_k)) \\ &= \sum_{k=1}^N c_k \cdot T(\chi_{[a_k, b_k]}) \\ &= T\left(\sum_{k=1}^N c_k \chi_{[a_k, b_k]}\right) \\ &\leq \|T\| \left\| \sum_{k=1}^N c_k \chi_{[a_k, b_k]} \right\|_p. \end{aligned}$$

By the disjointedness of the intervals, we may write

$$\int_a^b \left| \sum_{k=1}^N c_k \chi_{[a_k, b_k]} \right|^p dx \leq \sum_{k=1}^N \int_{a_k}^{b_k} dx = \sum_{k=1}^N (b_k - a_k).$$

So, $\left\| \sum_{k=1}^N c_k \chi_{[a_k, b_k]} \right\|_p = \left(\sum_{k=1}^N (b_k - a_k) \right)^{\frac{1}{p}}$, thus

$$\sum_{k=1}^N |G_T(b_k) - G_T(a_k)| \leq \|T\| \cdot \left(\sum_{k=1}^N (b_k - a_k) \right)^{\frac{1}{p}}.$$

Hence, for $\varepsilon > 0$, letting $\delta = \left(\frac{\varepsilon}{\|T\|} \right)^p$ proves absolute continuity of G_T . Thus, $g = G'_T$ exists and is such that $g \in L^1([a, b])$ and

$$G_T(x) = \int_a^x g(t) dt, \quad \forall x \in [a, b].$$

Hence,

$$\begin{aligned}
T(\chi_{[c,d]}) &= G_T(d) - G_T(c) = \int_a^d g(t) dt - \int_a^c g(t) dt \\
&= \int_c^d g(t) dt \\
&= \int_a^b g(t) \cdot \chi_{[c,d]}(t) dt.
\end{aligned}$$

This proves the theorem for indicator functions; by linearity of T and linearity of the integral, we can repeat this procedure to find a function g such that $Tf = \int_a^b f(t)g(t) dt$ for every step function f .

Step 2: Let f bounded and measurable. We know that for every step function ψ , $T\psi = \int_a^b \psi(t)g(t) dt$ (with the g as “found” in step 1). So,

$$\begin{aligned}
\left| Tf - \int_a^b f(t)g(t) dt \right| &= \left| T(f - \psi) - \int_a^b (f(t) - \psi(t))g(t) dt \right| \\
&\leq \|T\| \|f - \psi\|_p + \int_a^b |f(t) - \psi(t)| |g(t)| dt.
\end{aligned}$$

Then, since $g \in L^1([a, b])$, for every $\varepsilon > 0$ there is some $\delta > 0$ such that if E a set of measure less than δ , $\int_E |g(t)| dt < \varepsilon$. Fix $\varepsilon > 0$ and $\delta > 0$ such that this holds; let $\delta < \varepsilon$ if necessary wlog. Since f bounded and measurable, there is some step function ψ such that $|f - \psi| < \delta$ on $E \subseteq [a, b]$, and that $m(E^c) < \delta$ and $|\psi| \leq \|f\|_\infty$. Hence,

$$\begin{aligned}
\|f - \psi\|_p^p &= \int_E |f - \psi|^p + \int_{E^c} |f - \psi|^p \\
&\leq \delta^p \cdot m(E) + (2\|f\|_\infty)^p m(E^c) \\
&\leq \delta^p |b - a| + (2\|f\|_\infty)^p \delta.
\end{aligned}$$

Also,

$$\begin{aligned}
\int_a^b |f - \psi| |g| dt &\leq \int_E \delta \cdot |g| dt + \int_{E^c} 2\|f\|_\infty |g| dt \\
&\leq \delta \|g\|_1 + 2\|f\|_\infty \varepsilon.
\end{aligned}$$

All together then,

$$\begin{aligned}
\left| Tf - \int_a^b f(t)g(t) dt \right| &\leq \|T\| \left(\delta^p |b - a| + (2\|f\|_\infty)^p \delta \right)^{\frac{1}{p}} + \delta \|g\|_1 + 2\|f\|_\infty \varepsilon \\
&< C(\|f\|_\infty, \|g\|_1, a, b, \|T\|) \cdot \varepsilon^{\frac{1}{p}},
\end{aligned}$$

where C a constant. The LHS does not depend on ε , hence taking the limit $\varepsilon \rightarrow 0^+$, we conclude

$$Tf = \int_a^b f(t)g(t) dt.$$

Note that all simple functions are bounded and measurable, so the necessary property also holds for f simple.

We need now to show $g \in L^q([a, b])$ and $\|g\| = \|T\|$.

- Case 1: $p > 1$ so $q < \infty$. Let $g_n := \begin{cases} g & \text{if } |g| \leq n \\ 0 & \text{o.w.} \end{cases}$ and $f_n := \begin{cases} |g|^{q-1} \operatorname{sgn}(g) & \text{if } |g| \leq n \\ 0 & \text{o.w.} \end{cases}$. Then,

$$\begin{aligned} \|g_n\|_q^q &= \int_{\{|g| \leq n\}} |g|^q dt \\ &= \int_{\{|g| \leq n\}} f_n \cdot g_n dt \\ &= \int_{\{|g| \leq n\}} f_n g dt \\ &= Tf_n \leq \|T\| \|f_n\|_p, \end{aligned}$$

since f_n bounded and measurable so Step 2 applies. Also,

$$\begin{aligned} \|f_n\|_p^p &= \int_{\{|g| \leq n\}} |g|^{(q-1)p} dt \\ &= \int_{\{|g| \leq n\}} |g|^q dt = \|g_n\|_q^q. \end{aligned}$$

All together then,

$$\|g_n\|_q^q \leq \|T\| \|g_n\|_q^{q/p} \Rightarrow \|g_n\|_q^{q(1-\frac{1}{p})} = \|g_n\|_q \leq \|T\|.$$

By construction, $|g_n|^q \rightarrow |g|^q$ a.e. and monotonely, so by the monotone convergence theorem,

$$\|g_n\|_q \rightarrow \|g\|_q,$$

so $\|g\|_q \leq \|T\|$ and so $g \in L^q([a, b])$. From here, as in the example at the beginning of this section, one can show equality by choosing f appropriately.

- Case 2: $p = 1$ so $q = \infty$. We claim that $\|g\|_\infty = \sup_{\|f\|_1=1, f \text{ bdd}} \int fg$. Let $\varepsilon > 0$ and $A \subseteq [a, b]$ such that $|g| \geq \|g\|_\infty - \varepsilon$ on A where $m(A) > 0$. Let

$$f(x) = \frac{\chi_A}{m(A)} \operatorname{sgn}(g).$$

Then, f bounded and $\|f\|_1 = 1$. So,

$$\int fg = \frac{1}{m(A)} \int_A |g| \geq \frac{1}{m(A)} \int_A (\|g\|_\infty - \varepsilon) = \|g\|_\infty - \varepsilon,$$

hence we have proven \leq of our claim. By Holder,

$$\sup_{\|f\|=1} \int fg \leq \|f\|_1 \|g\|_\infty = \|g\|_\infty,$$

so \geq holds and the claim is proven. Thus,

$$\|g\|_\infty = \sup_{\substack{\|f\|=1, \\ f \text{ bdd}}} Tf \leq \|T\| \|f\|_1 = \|T\|,$$

so in particular $g \in L^\infty([a, b])$. For the other inequality,

$$|Tf| = \left| \int fg \, dt \right| \leq \|f\|_1 \|g\|_\infty,$$

hence

$$\|T\| \leq \|g\|_\infty$$

so $\|g\|_\infty = \|T\|$ as we aimed to show.

Step 3: We need to show $Tf = \int_a^b fg \, dt$ for every $f \in L^p([a, b])$. Simple functions are dense in $L^p([a, b])$, and since $Tf = \int_a^b fg \, dt$ for every simple function f , we conclude $Tf = \int_a^b fg \, dt$ for every $f \in L^p([a, b])$ by the previous density lemma.

Moreover, g is unique because if

$$\int_a^b fg = \int_a^b fg',$$

then

$$\int_a^b f(g - g') = 0,$$

for every $f \in L^p$. Let $f(t) = \text{sgn}(g - g')$, then

$$0 = \int_a^b |g - g'| \, dt \Rightarrow g = g' \text{ a.e..}$$

So, g uniquely defined up to a set of measure 0 so $g = g'$ in L^q . ■

PROOF (Of RRT if $\Omega = \mathbb{R}$). Fix $T \in (L^p(\mathbb{R}))^*$. Then, $T|_{[-N, N]} \in (L^p([-N, N]))^*$ for every $N \geq 1$, and $\|T|_{[-N, N]}\| \leq \|T\|$. Then, by RRT on $[-N, N]$, there is a $g_N \in L^q([-N, N])$ such that $Tf = \int_{-N}^N fg_N \, dt$. By uniqueness, $g_{N+1}|_{[-N, N]} = g_N$. Define

$$g(t) := g_N(t), \quad t \in [-N, N].$$

So, $g_N(t) \rightarrow g(t)$ pointwise and $|g_N(t)|^q \rightarrow |g(t)|^q$ pointwise and monotonely. By monotone convergence, then, $\int_{\mathbb{R}} |g_N|^q \, dt \rightarrow \int_{\mathbb{R}} |g|^q \, dt$. So, $g \in L^q(\mathbb{R})$ since

$\|g_N\|_{L^q([-N, N])} \leq \|T\|$ for every $N \geq 1$. Let $f_N(t) = f(t)\chi_{[-N, N]}$. Then, $f_N \rightarrow f$ in $L^p(\mathbb{R})$ so $Tf_N \rightarrow Tf$. So also

$$Tf_N = \int_{-N}^N f_N g_N = \int_{-N}^N f(t) g_N(t) \, dt = \int_{\mathbb{R}} f g_N \, dt \rightarrow Tf,$$

if we take by convention the g_N 's to be zero outside of $[-N, N]$. But also, $f \in L^p(\mathbb{R})$ and $g_N \rightarrow g$ in $L^q(\mathbb{R})$, so applying Holder's to the quantity $\int_{\mathbb{R}} f g_N$, we know

$$\int_{\mathbb{R}} f g_N \rightarrow \int_{\mathbb{R}} f g,$$

hence equating the two

$$Tf = \int_{\mathbb{R}} f g,$$

for every $f \in L^p(\mathbb{R})$. A similar proof to the previous gives the necessary norm identity. ■

PROOF (Of RRT for general $\Omega \subseteq \mathbb{R}$). If $T \in (L^p(\Omega))^*$, let $\hat{T} \in (L^p(\mathbb{R}))^*$ given by $\hat{T}f = T(f|_{\Omega})$. Then by the previous case there is $\hat{g} \in L^q(\mathbb{R})$ such that $\hat{T}(f) = \int f \hat{g}$. Let $g = \hat{g}|_{\Omega}$, then $Tf = \int_{\Omega} f g$. ■

So, RRT gives us that for $p \in [1, \infty]$, $(L^p(\Omega))^* \sim L^q(\Omega)$, and that $\|f\|_p = \sup_{\|g\|_q=1} |\int f g|$.

In particular, if $p = 1$,

$$\|f\|_{L^1} = \int f \operatorname{sgn} f(x) dx = \sup_{\|g\|_{\infty}=1} \int f g.$$

What, though, is $(L^{\infty})^*$. Certainly, $L^1(\Omega) \subseteq (L^{\infty}(\Omega))^*$ since for $f \in L^{\infty}$, $Tf = \int f g dx$ with $g \in L^1$, which is bounded by Holders. However, it turns out that this inclusion is a strict one. Consider for instance

$$Tf := f(0), \quad T : L^{\infty}([-1, 1]) \rightarrow \mathbb{R}.$$

Then, certainly $|Tf| \leq \|f\|_{\infty}$ so $T \in (L^{\infty})^*$. However, there is no function g such that $f(0) = \int f(t)g(t) dt$.

§2.10 Weak Convergence in $L^p(\Omega)$

↪ **Definition 2.16** (Weak convergence in $L^p(\Omega)$): Let $\Omega \subset \mathbb{R}^d$, $p \in [1, \infty)$ and q its conjugate. Then, we say $f_n \rightarrow f$ *weakly* in $L^p(\Omega)$, and write

$$f_n \xrightarrow{L^p(\Omega)} f,$$

if for every $g \in L^q(\Omega)$,

$$\lim_{n \rightarrow \infty} \int_{\Omega} f_n g dx = \int_{\Omega} f g dx.$$

Remark 2.18: Weak limits are unique; suppose otherwise that $f_n \rightharpoonup f, \bar{f}$. Let $g = \text{sgn}(f - \bar{f}) \cdot |f - \bar{f}|^{p-1}$, which is in $L^q(\Omega)$. So,

$$\lim_n \int g f_n \, dx = \int g f \, dx = \int g \bar{f} \, dx,$$

by assumption, so

$$0 = \int_{\Omega} g(f - \bar{f}) \, dx = \int |f - \bar{f}|^p \, dx,$$

hence $f = \bar{f}$ a.e. (and so equal as elements of $L^p(\Omega)$).

Remark 2.19: Many of the properties of weakly convergent sequences in a Hilbert space carry over to this setting.

↪ **Proposition 2.12:** Let $\Omega \subseteq \mathbb{R}^d$.

(i) If $p \in (1, \infty)$, $f_n \xrightarrow{L^p(\Omega)} f$, then $\{f_n\} \subseteq L^p(\Omega)$ are bounded, and moreover $\|f\|_p \leq \liminf_n \|f_n\|_p$.

(ii) If $p \in [1, \infty)$ and $f_n \xrightarrow{L^p(\Omega)} f, g_n \xrightarrow{L^p(\Omega)} g$, then $\lim_{n \rightarrow \infty} \int g_n f_n \, dx = \int g f \, dx$.

PROOF. Identical to Hilbert space proofs; replace usage of Cauchy-Schwarz with Holder's. ■

Remark 2.20: In (i), $p \in (1, \infty)$, since L^p “reflexive” in this case, i.e. $(L^p)^{**} = L^p$ (just as we had in the Hilbert space case). We don't have this property for $p = 1$.

Remark 2.21: A related notion of convergence is called *weak* convergence*, written $f_n \xrightarrow{L^p(\Omega)^*} f$; we say this holds if for every $g \in L^q(\Omega)$ such that $(L^q)^* = L^p$, then $\int f_n g \, dx \rightarrow \int f g \, dx$. So if $p \in (1, \infty)$, weak convergence = weak* convergence, by Riesz.

Remark 2.22: There are many equivalent notions to weak convergence.

↪ **Theorem 2.22** (Equivalent Weak Convergence): Let $p \in (1, \infty)$. Suppose $\{f_n\} \subseteq L^p(\Omega)$ are bounded and $f \in L^p$. Then, $f_n \xrightarrow{L^p(\Omega)} f$ iff

- for any $g \in G \subseteq L^q(\Omega)$ such that $\overline{\text{span}(G)} = L^q(\Omega)$, then $\lim_{n \rightarrow \infty} \int f_n g = \int f g$;
- $\forall A \subseteq \Omega$ measurable with finite measure, then $\lim_{n \rightarrow \infty} \int_A f_n \, dx = \int_A f \, dx$;
- if $d = 1$ and $\Omega = [a, b]$, then $\lim_{n \rightarrow \infty} \int_a^x f_n \, dx = \int_a^x f \, dx$ for every $x \in [a, b]$.
- $f_n \rightarrow f$ pointwise a.e..

Remark 2.23: Some of these notions extend to $p = 1$, but we state in the $p > 1$ case for simplicity.

↪ **Theorem 2.23** (Radon-Riesz): Let $p \in (1, \infty)$. Suppose $f_n \xrightarrow{L^p(\Omega)} f$, then $f_n \xrightarrow{L^p(\Omega)} f$ iff $\lim_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p$.

Alternatively, there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \rightarrow f$ in $L^p(\Omega)$ iff $\liminf_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p$.

PROOF. (\Rightarrow) If $f_n \xrightarrow{L^p(\Omega)} f$ then $\|f_n\|_p \rightarrow \|f\|_p$ by triangle inequality.

The converse, (\Leftarrow), is hard. ■

↪ **Theorem 2.24** (Weak Compactness): Let $p \in (1, \infty)$, then every bounded sequence in $L^p(\Omega)$ has a weakly convergent subsequence, with limit in $L^p(\Omega)$.

PROOF. Let $\{f_n\} \subseteq L^p(\Omega)$ be bounded. $p \in (1, \infty)$ so so is q , and in particular $L^q(\Omega)$ is separable. Let $T_n \in (L^q(\Omega))^*$ be given by $T_n(g) := \int f_n g \, dx$ for $g \in L^q(\Omega)$. Then, $\|T_n\| = \|f_n\|_p \leq C$. So,

$$\sup_n |T_n(g)| \leq \|T_n\| \|g\|_q \leq C \|g\|_q.$$

By Helley's Theorem (Thm. 2.14), there exists a subsequence $\{T_{n_k}\}$ and $T \subseteq (L^q(\Omega))^*$ such that $\lim_{k \rightarrow \infty} T_{n_k}(g) = T(g)$ for every $g \in L^q(\Omega)$. By Riesz, there exists some $f \in L^p(\Omega)$ such that $T(g) = \int f g \, dx$, and hence

$$\lim_k \int f_{n_k} g \, dx = \int f g \, dx,$$

for every $g \in L^q(\Omega)$, so $f_{n_k} \xrightarrow{L^p(\Omega)} f$. ■

§2.11 Convolution and Mollifiers

↪ **Definition 2.17** (Convolution):

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x - y)g(y) \, dy = \int_{\mathbb{R}^d} f(y)g(x - y) \, dy.$$

↪ **Proposition 2.13** (Properties of Convolution):

- a. $(f * g) * h = f * (g * h)$ (convolution is associative)
- b. Let $\tau_z f(x) := f(x - z)$ be the z -translate of f which centers f at z . Then,

$$\tau_z(f * g) = (\tau_z f) * g = f * (\tau_z g).$$

- c. $\text{supp}(f * g) \subseteq \overline{\{x + y \mid x \in \text{supp}(f), y \in \text{supp}(g)\}}$.

PROOF. (a) Assuming all the necessary integrals are finite, we can change order of integration,

$$\begin{aligned} ((f * g) * h)(x) &= \left(\int f(y)g(x - y) \, dy \right) * h(x) \\ &= \int \int f(y)g(x - z - y) \, dy, h(z) \, dz \\ &= \int \int f(y)g(x - y - z)h(z) \, dz \, dy \quad (y' = x - y) \\ &= \int \int f(x - y')g(y' - z)h(z) \, dz \, dy' \\ &= \int f(x - y')(g * h)(y') \, dy' = (f * (g * h))(x). \end{aligned}$$

(b) For the first equality,

$$\begin{aligned} \tau_z(f * g)(x) &= \tau_z \int f(x - y)g(y) \, dy \\ &= \int f(x - z - y)g(y) \, dy \\ &= \int (\tau_z f(x - y))g(y) \, dy = ((\tau_z f) * g)(x). \end{aligned}$$

The second follows from a change of variables in the second line.

(c) We'll show that $A^c \subseteq (\text{supp}(f * g))^c$ where A the set as defined in the proposition. Let $x \in A^c$, then if $y \in \text{supp}(g)$, $x - y \notin \text{supp}(f)$ so $f(x - y) = 0$; else if $y \notin \text{supp}(g)$ it must be $g(y) = 0$. So, if $x \in A^c$, it must be that

$$\int f(x - y)g(y) \, dy = \int_{\text{supp}(g)} \underbrace{f(x - y)}_{=0} g(y) \, dy + \int_{\text{supp}(g)^c} f(x - y) \underbrace{g(y)}_{=0} \, dy = 0.$$

■

We've been rather loose with finiteness of the convolutions so far. To establish this, we need the following result.

↪ **Theorem 2.25** (Young's Inequality): Let $f \in L^1(\mathbb{R}^d)$, $g \in L^p(\mathbb{R}^d)$ for any $p \in [1, \infty]$. Then,

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p,$$

hence $f * g \in L^p(\mathbb{R}^d)$.

PROOF. Suppose first $p = \infty$, then

$$(f * g)(x) = \int f(y)g(x-y) dy \leq \|g\|_\infty \int |f(y)| dy = \|g\|_\infty \|f\|_1,$$

for every $x \in \mathbb{R}^d$, so passing to the L^∞ -norm,

$$\|f * g\|_\infty \leq \|f\|_1 \|g\|_\infty.$$

Suppose now $p = 1$. Then,

$$\|f * g\|_1 = \int \left| \int f(x-y)g(y) dy \right| dx.$$

Let $F(x, y) = f(x-y)g(y)$, then for almost every $y \in \mathbb{R}^d$,

$$\begin{aligned} \int |F(x, y)| dx &= \int |g(y)| |f(x-y)| dx \\ &= |g(y)| \int |f(x-y)| dx \\ &= |g(y)| \|f\|_1. \end{aligned}$$

Applying Tonelli's Theorem, we have then

$$\iint |F(x, y)| dy dx = \iint |F(x, y)| dx dy = \int |g(y)| \|f\|_1 dy = \|f\|_1 \|g\|_1,$$

(so really $F \in L^1(\mathbb{R}^d) \times L^1(\mathbb{R}^d)$), hence all together

$$\|f * g\|_1 = \int \left| \int F(x, y) dy \right| dx \leq \iint |F(x, y)| dy dx = \|f\|_1 \|g\|_1.$$

Remark 2.24: It also follows that for a.e. $x \in \mathbb{R}^d$, $\int |F(x, y)| dy < \infty$, i.e. $\int |f(x-y)g(y)| dy < \infty$. Moreover, since if $g \in L^p(\Omega)$ then $|g|^p \in L^1(\Omega)$, a similar argument gives that for almost every $x \in \mathbb{R}^d$, $\int |f(x-y)||g(y)|^p dy < \infty$.

Suppose now $1 < p < \infty$. For a.e. $x \in \mathbb{R}^d$, $\int |g(y)|^p |f(x-y)| dy < \infty$, so $g \in L^p(\mathbb{R}^d)$ implies for a.e. $x \in \mathbb{R}^d$, $|g(\cdot)|^p |f(x-\cdot)| \in L^1(\mathbb{R}^d)$ as a function of \cdot . This further implies $g(y)f^{\frac{1}{p}}(x-y) \in L^p(\mathbb{R}^d, dy)$. Also, if $f \in L^1(\mathbb{R}^d)$, then $f^{\frac{1}{q}} \in L^q(\mathbb{R}^d)$. All together then,

$$\int |f(x-y)| |g(y)| dy = \int \underbrace{\left| f^{\frac{1}{q}}(x-y) \right|}_{\substack{q \\ p}} \underbrace{\left| f^{\frac{1}{p}}(x-y) \right| |g(y)|}_{p} dy$$

$$\text{Holder's} \leq \left(\int |f(x-y)| dy \right)^{\frac{1}{q}} \left(\int |f(x-y)| |g(y)|^p dy \right)^{\frac{1}{p}},$$

hence, raising both sides to the p ,

$$|(f * g)(x)|^p \leq \|f\|_1^{\frac{p}{q}} \cdot (|f| * |g|^p)(x)$$

and integrating both sides

$$\int |(f * g)(x)|^p dx \leq \|f\|_1^{\frac{p}{q}} \int \left(\underbrace{|f|}_{\in L^1(\mathbb{R}^d)} * \underbrace{|g|^p}_{\in L^1(\mathbb{R}^d)} \right)(x) dx.$$

Hence, we can bound the right-hand term using the previous case for $p = 1$, and find

$$\begin{aligned} \int |(f * g)(x)|^p dx &\leq \|f\|_1^{\frac{p}{q}} \|f\|_1 \|g^p\|_1 \\ &= \|f\|_1^{\frac{p}{q}+1} \|g\|_p^p \\ &= \|f\|_1^{\frac{p+q}{q}} \|g\|_p^p \\ \left(\frac{p+q}{q} = p \right) &= \|f\|_1^p \|g\|_p^p, \end{aligned}$$

so raising both sides to $\frac{1}{p}$, we conclude

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

■

↪ **Proposition 2.14:** If $f \in L^1(\mathbb{R}^d)$ and $g \in C^1(\mathbb{R}^d)$ with $|\partial_{x_i} g| \in L^\infty(\mathbb{R}^d)$ for $i = 1, \dots, d$, then $(f * g) \in C^1(\mathbb{R}^d)$ and moreover

$$\partial_{x_i}(f * g) = f * (\partial_{x_i} g).$$

Remark 2.25: There are many different conditions we can place on f, g to make this true; most basically, we need $|(\partial_i g) * f| < \infty$.

PROOF.

$$\frac{\partial}{\partial x_i} \left(\int f(y) g(x-y) dy \right) = \int \underbrace{f(y)}_{\in L^1(\mathbb{R}^d)} \underbrace{\partial_i g(x-y)}_{\in L^\infty(\mathbb{R}^d)} dy < \infty,$$

citing the previous theorem for the finiteness; the dominated convergence theorem allows us to pass the derivative inside. ■

Remark 2.26: This also follows for the gradient; namely $\nabla(f * g) = f * (\nabla g)$ with a component-wise convolution.

Consider the function

$$\rho(x) = \begin{cases} C \exp\left(-\frac{1}{1-|x|^2}\right) & \text{if } |x| \leq 1, \\ 0 & \text{o.w.} \end{cases}$$

where $C = C(d)$ a constant such that $\int_{\mathbb{R}^d} \rho(x) dx = 1$. Then, note that $\rho \in C_c^\infty(\mathbb{R}^d)$ (infinitely differentiable with compact support). Let now

$$\rho_\varepsilon(x) := \frac{1}{\varepsilon^d} \rho\left(\frac{x}{\varepsilon}\right).$$

Notice that $\rho_\varepsilon(x)$ is supported on $B(0, \varepsilon)$, but

$$\int_{\mathbb{R}^d} \rho_\varepsilon(x) dx = \frac{1}{\varepsilon^d} \int_{\mathbb{R}^d} \rho\left(\frac{x}{\varepsilon}\right) dx = \frac{1}{\varepsilon^d} \cdot \varepsilon^d \cdot \int_{\mathbb{R}^d} \rho(y) dy = 1,$$

for every ε , by making a change of variables $y = \frac{x}{\varepsilon}$. We'll be interested in the convolution

$$f_\varepsilon(x) := (\rho_\varepsilon * f)(x)$$

for some function f . ρ_ε is often called a “convolution kernel”. In particular, it is a “good kernel”, namely has the properties:

- $\int_{\mathbb{R}^d} \rho_\varepsilon(y) dy = 1$;
- $\int_{\mathbb{R}^d} |\rho_\varepsilon(y)| dy \leq M$ for some finite M ;
- $\forall \delta > 0, \int_{\{|y|>\delta\}} |\rho_\varepsilon(y)| dy \xrightarrow{\varepsilon \rightarrow 0} 0$.

The second condition is trivially satisfied in this case since our kernel is nonnegative. The last also follows easily since ρ_ε has compact support; more generally, this imposes rapid decay conditions on the tails of good kernels.

Since $\rho_\varepsilon \in C_c^\infty(\mathbb{R}^d)$, for “reasonable” f , $f_\varepsilon = \rho_\varepsilon * f \in C^\infty(\mathbb{R}^d)$ by the previous proposition. In fact, we'll see that in many contexts $f_\varepsilon \rightarrow f$ as $\varepsilon \rightarrow 0$ in some notion of convergence. So, f_ε provides a good, now smooth, approximation to f .

Proposition 2.15: Suppose $f \in L^\infty(\mathbb{R}^d)$ and f_ε is well-defined. Then, if f is continuous at x , then $f_\varepsilon(x) \rightarrow f(x)$ as $\varepsilon \rightarrow 0$.

If $f \in C(\mathbb{R}^d)$, then $f_\varepsilon \rightarrow f$ uniformly on compact sets.

PROOF. f continuous at x gives that for every $\eta > 0$ there exists a $\delta > 0$ such that $|f(y) - f(x)| < \eta$ whenever $|x - y| < \delta$. Then

$$\begin{aligned}
|f_\varepsilon(x) - f(x)| &= \left| \int \rho_\varepsilon(y) f(x-y) dy - f(x) \underbrace{\int \rho_\varepsilon(y) dy}_{=1} \right| \\
&= \left| \int \rho_\varepsilon(y) (f(x-y) - f(x)) dy \right| \\
&\leq \int_{\{|y| \leq \delta\}} |f(x-y) - f(x)| |\rho_\varepsilon(y)| dy + \int_{\{|y| > \delta\}} |f(x-y) - f(x)| |\rho_\varepsilon(y)| dy \\
&\stackrel{\substack{\text{(cnty in first argument)} \\ \text{(} L^\infty\text{-bound in second)}}}{\leq} \int_{\{|y| \leq \delta\}} \eta |\rho_\varepsilon(y)| dy + 2\|f\|_\infty \int_{\{|y| > \delta\}} |\rho_\varepsilon(y)| dy \\
&\leq \eta \cdot M + 2\|f\|_\infty \int_{\{|y| > \delta\}} |\rho_\varepsilon|
\end{aligned}$$

for $\varepsilon \rightarrow 0$, by using the second property of good kernels for the first bound. By the last property, the right-most term $\rightarrow 0$ as $\varepsilon \rightarrow 0$; moreover, then,

$$\lim_{\varepsilon \rightarrow 0} |f_\varepsilon(x) - f(x)| \leq C\eta$$

for some C and every $\eta > 0$, and thus $f_\varepsilon(x) \rightarrow f(x)$ as $\varepsilon \rightarrow 0$.

Now, if $f \in C(\mathbb{R}^d)$ fix a subset $K \subseteq \mathbb{R}^d$ compact. Hence, $\|f\|_{L^\infty(K)} < \infty$ and f uniformly continuous on K since K compact; so the modulus of continuity is uniform for all $x \in K$, so for $\delta > 0$ and for every $x \in K$,

$$\int_{\{|y| \leq \delta\}} |f(x-y) - f(x)| |\rho_\varepsilon(y)| dy \leq C\eta.$$

Also, using the bound on f , we may write the second integral in the argument above as

$$\int_{\varepsilon > |y| > \delta} |f(x-y) - f(x)| |\rho_\varepsilon(y)| dy \leq \|f\|_{L^\infty(K+B_\varepsilon)} \int_{\{|y| > \delta\}} |\rho_\varepsilon(y)| dy \xrightarrow{\varepsilon \rightarrow 0} 0$$

where we take K slightly larger as $K + B_\varepsilon$, which is still compact. So, since this held for all $x \in K$,

$$\max_{x \in K} |f_\varepsilon(x) - f(x)| \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Note that we proved the first for general good kernels but the second only in our constructed one. ■

Remark 2.27: This pointwise convergence result is why “good kernels” are called “approximations to the identity”.

Remark 2.28: If $f \in C_c(\mathbb{R}^d)$, then $\text{supp}(f_\varepsilon) \subseteq \overline{\text{supp}(f) + B(0, \varepsilon)}$; so, f_ε is compactly supported if f is. Hence in this case $f_\varepsilon \rightarrow f$ uniformly on \mathbb{R}^d . More generally, there are many different restrictions one can place on the last claim, such as compact support of f , uniform continuity of f , compact support of the kernel, lack of compact support for the kernel but an L^∞ bound on f , etc. In practice, the proofs are all the same, with different bounds; namely one finds something of the form

$$|f_\varepsilon(x) - f(x)| \leq \underbrace{\int_{|y| < \delta} (\dots)}_{\text{small by (uniform) continuity}} + \underbrace{\int_{|y| \geq \delta} (\dots)}_{\text{small by compact support, etc}}$$

↪ **Theorem 2.26** (Weierstrass Approximation Theorem): Let $[a, b] \subseteq \mathbb{R}$ and let $f \in C([a, b])$. Then for every $\eta > 0$, there exists a polynomial $P_N(x)$ of degree N such that

$$\|P_N - f\|_{L^\infty([a, b])} < \eta.$$

That is, polynomials are dense in $C([a, b])$.

PROOF. Extend f to be continuous with compact support on all of \mathbb{R} in whatever convenient way, such that $\text{supp}(f) \subseteq [-M, M]$ for some sufficiently large $M > 0$. Consider now

$$K_\varepsilon(x) := \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi x^2}{\varepsilon}},$$

noting that

$$\int_{-\infty}^{\infty} K_\varepsilon(x) dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi x^2}{\varepsilon}} dx = 1,$$

which is clear by a change of variables $y = \frac{\sqrt{2\pi}}{\sqrt{\varepsilon}} x$. As a consequence, $\int_{-\infty}^{\infty} |K_\varepsilon(x)| dx = 1 < \infty$, since $K_\varepsilon \geq 0$. Finally,

$$\begin{aligned} \int_{|x| > \delta} |K_\varepsilon(x)| dx &= \int_{|x| > \delta} \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi x^2}{\varepsilon}} dx \\ &= \int_{|y| > \frac{\sqrt{2\pi}}{\sqrt{\varepsilon}} \delta} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\ &\leq \int_{|y| > \frac{\sqrt{2\pi}}{\sqrt{\varepsilon}} \delta} \frac{|y|}{\sqrt{2\pi}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy \\ &\leq C e^{-\frac{y^2}{2}} \Big|_{\frac{\sqrt{2\pi}}{\sqrt{\varepsilon}} \delta}^{\infty} \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

since $|y| \geq 1$ here for suff. small ε

So, K_ε is a good kernel, and so $(f * K_\varepsilon)(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} f$ uniformly in $[a, b]$ by our last remark. In particular, for $\eta > 0$ there is some $\varepsilon_0 > 0$,

$$\|(f * K_{\varepsilon_0}) - f\|_{L^\infty([a,b])} < \frac{\eta}{2}.$$

We claim now that there is a polynomial P_N such that $\|P_N - (f * K_{\varepsilon_0})\|_{L^\infty([a,b])} < \frac{\eta}{2}$. Recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, which converges uniformly on compact sets. So, there exists a polynomial S_N (from truncating this sum) such that $\|K_{\varepsilon_0} - S_N\|_{L^\infty([-M,M])} < \frac{\eta}{4\|f\|_\infty M}$. Thus,

$$\begin{aligned} |f * K_{\varepsilon_0}(x) - f * S_N(x)| &\leq \left| \int f(x-y)(K_{\varepsilon_0}(y) - S_N(y)) dy \right| \\ \text{supp}(f) \subset [-M, M] &\leq \int_{-M}^M |f(x-y)| |K_{\varepsilon_0}(y) - S_N(y)| dy \\ &\leq 2M\|f\|_\infty \frac{\eta}{4M\|f\|_\infty} = \frac{\eta}{2}, \end{aligned}$$

for every x . Let $P_N(x) = (f * S_N)(x)$, which we see to be a polynomial. ■

→ Theorem 2.27: Let $f \in L^p(\mathbb{R}^d)$ with $p \in [1, \infty)$. Then $f_\varepsilon \xrightarrow{L^p(\mathbb{R}^d)} f$.

PROOF. Since $f \in L^p(\mathbb{R}^d)$, for every $\eta > 0$ there is a $\tilde{f} \in C_c(\mathbb{R}^d)$ such that $\|f - \tilde{f}\|_p < \eta$. Since $\tilde{f} \in C_c(\mathbb{R}^d)$, by the previous theorem dealing with mollifiers and uniform convergence, $\tilde{f}_\varepsilon \rightarrow \tilde{f}$ uniformly. In particular, we have $\|\tilde{f}_\varepsilon - \tilde{f}\|_p \xrightarrow{p} 0$, hence

$$\|f - f_\varepsilon\|_p \leq \|f_\varepsilon - \tilde{f}_\varepsilon\|_p + \|\tilde{f}_\varepsilon - \tilde{f}\|_p + \|\tilde{f} - f\|_p.$$

We've dealt with the second two bounds. For the first,

$$\begin{aligned} \|f_\varepsilon - \tilde{f}_\varepsilon\|_p &= \|(f - \tilde{f}) * \rho_\varepsilon\|_p \\ (\text{Young's}) &\leq \|\rho_\varepsilon\|_1 \|f - \tilde{f}\|_p = \|f - \tilde{f}\|_p, \end{aligned}$$

so

$$\|f - f_\varepsilon\|_p \leq 2\|f - \tilde{f}\|_p + \|\tilde{f}_\varepsilon - \tilde{f}\|_p < 3\eta.$$

■

→ Corollary 2.6: $C_c^\infty(\mathbb{R}^d)$ dense in $L^p(\mathbb{R}^d)$.

PROOF. We showed \tilde{f}_ε approximates f in $L^p(\mathbb{R}^d)$, and by construction \tilde{f}_ε is smooth with compact support. ■

§2.12 Strong Compactness in $L^p(\mathbb{R}^d)$

We saw that for $p \in (1, \infty)$, $\{f_n\} \subset L^p(\Omega)$, that any bounded sequence admits a weakly converging subsequence, $f_{n_k} \xrightarrow{L^p} f$. In addition, if the norms also converge i.e. $\lim_{n \rightarrow \infty} \|f_n\|_p = \|f\|_p$, then we actually have strong convergence $f_{n_k} \xrightarrow{L^p} f$.

We provide now a strong compactness result in L^p , akin to Arzelà-Ascoli.

↪ **Theorem 2.28** (Strong Compactness): Let $\{f_n\} \subseteq L^p(\mathbb{R}^d)$ for $p \in [1, \infty)$ s.t.

- i. $\exists C > 0$ s.t. $\|f_n\|_p < C \forall n$, i.e. $\{f_n\}$ uniformly bounded in L^p ;
- ii. $\lim_{|h| \rightarrow 0} \|f_n - \tau_h f_n\|_p = 0$ uniformly in n , i.e. for every $\eta > 0$, there exists $\delta > 0$ such that if $|h| < \delta$, $\int |f_n(x) - f_n(x-h)|^p dx < \eta^p$ for every n ;

Then, for any $\Omega \subseteq \mathbb{R}^d$ with finite measure, there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \xrightarrow{L^p(\Omega)} f$.

PROOF. Recall that $L^p(\Omega)$ is a complete metric space, so TFAE:

1. sequential compactness;
2. totally bounded (& complete);
3. compact.

Let $\mathcal{F} = \{f \in L^p(\mathbb{R}^d) \text{ satisfying i., ii.}\}$ and fix $\Omega \subseteq \mathbb{R}^d$ with finite measure. We aim to show that $\mathcal{F}|_\Omega$ is sequentially compact in $L^p(\Omega)$ (with no regard to whether the limit lives in \mathcal{F}_Ω); equivalently, we wish to show $\mathcal{F}|_\Omega$ is precompact in $L^p(\Omega)$ i.e. $\overline{\mathcal{F}|_\Omega}$ is compact. Since $\overline{\mathcal{F}|_\Omega}$ is a complete metric space, to prove the result it suffices to show that $\mathcal{F}|_\Omega$ is totally bounded (recall: for every $\delta > 0$, $\mathcal{F}|_\Omega \subseteq \bigcup_{i=1}^N B_{L^p(\Omega)}(g_i, \delta)$). We'll do this using mollifiers and AA.

Step 1: Fix η, δ as in ii. in the statement of the theorem, and let $f \in \mathcal{F}$. Then, for every $\varepsilon < \delta$, we claim

$$\|(\rho_\varepsilon * f) - f\|_{L^p(\mathbb{R}^d)} < \eta.$$

We have

$$\begin{aligned} |(\rho_\varepsilon * f)(x) - f(x)| &= \left| \int_{B_\varepsilon} \rho_\varepsilon(y) f(x-y) dy - f(x) \int \rho_\varepsilon(y) dy \right| \\ &\leq \int_{B_\varepsilon} \rho_\varepsilon(y) |f(x-y) - f(x)| dy \\ &= \int_{B_\varepsilon} \rho_\varepsilon^{\frac{1}{q}}(y) \rho_\varepsilon^{\frac{1}{p}}(y) |f(x-y) - f(x)| dy \\ (\text{Holder's}) \quad &\leq \left(\int \rho_\varepsilon(y) |f(x-y) - f(x)|^p dy \right)^{1/p} \underbrace{\left(\int \rho_\varepsilon(y) dy \right)^{1/q}}_{=1}, \end{aligned}$$

and hence

$$\begin{aligned}
\int |(\rho_\varepsilon * f)(x) - f(x)|^p dx &\leq \iint \rho_\varepsilon(y) |f(x-y) - f(x)|^p dy dx \\
\text{(Tonelli's)} \quad &= \int_{B_\varepsilon} \rho_\varepsilon(y) \underbrace{\int |f(x-y) - f(x)|^p dx}_{\varepsilon < \delta \Rightarrow \eta^p} dy \\
&< \eta^p \underbrace{\int_{B_\varepsilon} \rho_\varepsilon(y) dy}_{=1} = \eta^p,
\end{aligned}$$

hence $\|(\rho_\varepsilon * f)(x) - f(x)\|_p < \eta$.

Step 2: We first claim that there exists some C_ε such that for any $f \in \mathcal{F}$,

$$\|\rho_\varepsilon * f\|_\infty \leq C_\varepsilon \|f\|_p, \quad (1)$$

and that for any $x_1, x_2 \in \mathbb{R}^d$,

$$|(\rho_\varepsilon * f)(x_1) - (\rho_\varepsilon * f)(x_2)| \leq C_\varepsilon \|f\|_p |x_1 - x_2|. \quad (2)$$

In particular, this shows that *for ε fixed*, $(\rho_\varepsilon * f)$ satisfy hypothesis of AA. Remark that the first is a uniform boundedness type condition for $\rho_\varepsilon * f$, and the second is a uniform Lipschitz bound.

For the first claim (1),

$$\begin{aligned}
|(\rho_\varepsilon * f)(x)| &= \left| \int \rho_\varepsilon(x-y) f(y) dy \right| \\
\text{(Holder's)} \quad &\leq \left(\int \rho_\varepsilon^q(x-y) dy \right)^{\frac{1}{q}} \cdot \|f\|_p \\
&= \|\rho_\varepsilon\|_q \|f\|_p,
\end{aligned}$$

so we have the bound with $C_\varepsilon := \|\rho_\varepsilon\|_q$ since the bound is independent of x .

Remark 2.29: One can explicitly compute $\|\rho_\varepsilon\|_{q'}$ and realize that it will in general depend explicitly on ε .

For the second statement (2), we find that $\nabla(\rho_\varepsilon * f) = (\nabla \rho_\varepsilon) * f$ since the RHS is finite, because

$$(\nabla \rho_\varepsilon * f)(x) = \int \nabla \rho_\varepsilon(x-y) f(y) dy \leq \|\nabla \rho_\varepsilon\|_q \|f\|_p.$$

So,

$$\|\nabla(\rho_\varepsilon * f)\|_\infty \leq \underbrace{\|\nabla \rho_\varepsilon\|_q}_{=: C_\varepsilon} \|f\|_p.$$

By the mean-value theorem then, we have all together

$$\begin{aligned}\|(\rho_\varepsilon * f)(x_1) - (\rho_\varepsilon * f)(x_2)\| &\leq \|\nabla(\rho_\varepsilon * f)\|_\infty |x_1 - x_2| \\ &\leq C_\varepsilon \|f\|_p |x_1 - x_2|.\end{aligned}$$

This proves (2).

Step 3: Next, we claim that for $\eta > 0$ and fixed $\varepsilon < \eta$ and $\Omega \subseteq \mathbb{R}^d$ with finite measure, there exists $E \subseteq \Omega \subseteq \mathbb{R}^d$ such that E is bounded, i.e. $E \subseteq B(0, M)$ where M sufficiently large, and moreover that $\|f\|_{L^p(\Omega \setminus E)} < \eta$ for every $f \in \mathcal{F}$.

We have that

$$\|f\|_{L^p(\Omega \setminus E)} \leq \|f - (\rho_\varepsilon * f)\|_{L^p(\mathbb{R}^d)} + \|\rho_\varepsilon * f\|_{L^p(\Omega \setminus E)}.$$

By the very first step of the proof, the first term is $< \eta$, so this is bounded by

$$\begin{aligned}&< \eta + \left(\int_{\Omega \setminus E} |\rho_\varepsilon * f|^p dx \right)^{1/p} \\ &< \eta + \|\rho_\varepsilon * f\|_\infty |\Omega \setminus E|^{\frac{1}{p}} \\ &< \eta + C_\varepsilon \|f\|_p |\Omega \setminus E|^{\frac{1}{p}}.\end{aligned}$$

C_ε finite and $\|f\|_p$ upper bounded uniformly over \mathcal{F} , so it suffices to construct E with the measure of $\Omega \setminus E$ sufficiently small, so we can get $\|f\|_{L^p(\Omega \setminus E)} < 2\eta$.

Step 4: Fix $\eta > 0$. We claim $\mathcal{F}|_\Omega$ is totally bounded. Let $\varepsilon < \delta$ then let

$$\mathcal{H} := (\rho_\varepsilon * \mathcal{F})|_{\overline{E}} = \{\rho_\varepsilon * f|_E : f \in \mathcal{F}\}.$$

$E \subseteq \Omega \subseteq \mathbb{R}^d$ is bounded implies \overline{E} is compact. So by Step 2., we showed $(\rho_\varepsilon * \mathcal{F})$ satisfies hypotheses of AA on \overline{E} . Hence, \mathcal{H} is precompact in $C(\overline{E})$. Thus, since we have uniform convergence we certainly have L^p convergence thus \mathcal{H} also precompact in $L^p(\overline{E})$. Thus, for $\eta > 0$, there exists $\{\overline{g}_i\} \subseteq L^p(\overline{E})$ such that

$$\mathcal{H} \subseteq \bigcup_{i=1}^N B_{L^p(\overline{E})}(\overline{g}_i, \eta). \quad \star$$

Let $g_i : \Omega \rightarrow \mathbb{R}$ be given by

$$g_i(x) = \begin{cases} \overline{g}_i & \text{on } E \\ 0 & \text{on } \Omega \setminus E \end{cases}.$$

Then, we claim $\mathcal{F}|_\Omega \subseteq \bigcup_{i=1}^N B_{L^p(\Omega)}(g_i, 3\eta)$. If $f \in \mathcal{F}$ by \star , there is an i such that $\|\rho_\varepsilon * f - \overline{g}_i\|_{L^p(\overline{E})} < \eta$. But also,

$$\begin{aligned}
\|f - g_i\|_{L^p(\Omega)}^p &= \int_{\Omega \setminus E} |f|^p dx + \int_{\overline{E}} |f - \bar{g}_i|^p dx \\
&= \|f\|_{L^p(\Omega \setminus E)}^p + \int_{\overline{E}} |f - \bar{g}_i|^p dx \\
(\text{Step 3.}) \quad &\leq \eta^p + \int_{\overline{E}} |f - \bar{g}_i|^p dx.
\end{aligned}$$

Recall $(a + b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}$. Applying this bound to the above, we find

$$\begin{aligned}
\|f - g_i\|_{L^p(\Omega)} &\leq \eta + \|f - \bar{g}_i\|_{L^p(\overline{E})} \\
&\leq \underbrace{\eta + \|f - f * \rho_\varepsilon\|_{L^p(\mathbb{R}^d)}}_{< \eta \text{ by Step 1.}} + \underbrace{\|(f * \rho_\varepsilon) - \bar{g}_i\|_{L^p(\overline{E})}}_{< \eta \text{ by } \star} \\
&\leq 3\eta.
\end{aligned}$$

Hence, $\mathcal{F}|_\Omega \subseteq \bigcup_{i=1}^N B(g_i, 3\eta)$, thus $\mathcal{F}|_\Omega$ is sequentially compact so any sequence in \mathcal{F} has a converging subsequence, which proves the theorem. ■

Remark 2.30: This can be extended to $L^p(\mathbb{R}^d)$ with some conditions.

§3 INTRODUCTION TO FOURIER ANALYSIS

References are Folland, Chapter 8 and *Fourier Analysis* by Stein & Sharkarchi.

§3.1 Fourier Series

We will denote the torus $\mathbb{T} = [0, 1) \simeq \mathbb{R}/\mathbb{Z}$ (with 1 identified back with 0), and specifically complex-valued functions on the torus

$$L^2(\mathbb{T}) = \left\{ f : \mathbb{T} \rightarrow \mathbb{C} \mid \int_0^1 |f(x)|^2 dx < \infty \right\},$$

where now $|\cdot|$ the modulus (i.e. $|a + bi|^2 = a^2 + b^2$). Equivalently, $f : \mathbb{T} \rightarrow \mathbb{C}$ can be identified with $\tilde{f} : \mathbb{R} \rightarrow \mathbb{C}$ which is periodic.

↪ **Proposition 3.1:** The function $L^2(\mathbb{T}) \times L^2(\mathbb{T}) \rightarrow \mathbb{C}$

$$(f, g) = \int_0^1 f(x) \overline{g(x)} dx$$

is an inner product on $L^2(\mathbb{T})$. In particular, $(L^2(\mathbb{T}), (\cdot, \cdot))$ a Hilbert space.

PROOF. For \mathbb{C} -valued functions, we need to check:

- linearity in the first variable: for $\alpha \in \mathbb{C}$,

$$(\alpha f + h, g) = \int_0^1 (\alpha f + h) \bar{g} dx = \alpha(f, g) + (h, g)$$

by linearity of the integral;

- conjugate symmetry:

$$\begin{aligned}
\int_0^1 f(x) \overline{g(x)} \, dx &= \int_0^1 (\operatorname{Re}(f) + i\operatorname{Im}(f))(\operatorname{Re}(g) - i\operatorname{Im}(g)) \, dx \\
&=: \int_0^1 (a + ib)(c - id) \, dx \\
&= \int_0^1 (ac + bd) + i(bc - ad) \, dx \\
&= \int_0^1 (ac + bd) - i(ad - bc) \, dx \\
&= \overline{\int_0^1 g \overline{f} \, dx} = \overline{(g, f)};
\end{aligned}$$

- f inner product with f properties:

$$(f, f) = \int_0^1 f(x) \overline{f(x)} \, dx = \int_0^1 |f(x)|^2 \, dx = \|f\|_{L^2(\mathbb{T})}^2 \geq 0, = 0 \text{ iff } f \equiv 0.$$

We know $L^2(\mathbb{T})$ is complete, so $L^2(\mathbb{T})$ a Hilbert space with this inner product since it induces the same norm as the usual norm L^2 -norm. ■

↪ **Theorem 3.1:** Let $e_n(x) := e^{2\pi i n x}$ for $n \in \mathbb{Z}$. Then, $\{e_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{T})$.

PROOF. For orthonormality, if $n \neq m$,

$$\begin{aligned}
(e_n, e_m) &= \int_0^1 e^{2\pi i n x} e^{-2\pi i m x} \, dx \\
&= \int_0^1 e^{2\pi i (n-m)x} \, dx \\
&= \frac{1}{2\pi i (n-m)} e^{2\pi i (n-m)x} \Big|_0^1 \\
&= \frac{1}{2\pi i (n-m)} [e^{2\pi i (n-m)} - 1] \\
&= \frac{1}{2\pi i (n-m)} \left[\underbrace{\cos(2\pi(n-m))}_{=1} + \underbrace{i \sin(2\pi(n-m))}_{=0} - 1 \right] = 0,
\end{aligned}$$

and if $n = m$,

$$(e_n, e_n) = \int_0^1 |e^{2\pi i n x}|^2 \, dx = \int_0^1 1 \, dx = 1.$$

To prove its a basis, we use Stone-Weierstrass. \mathbb{T} is compact; let

$$\mathcal{A} := \left\{ \sum_{n=-N}^N \alpha_n e_n : \alpha_n \in \mathbb{C}, N \in \mathbb{N} \right\}.$$

Notice $e_n e_m = e^{2\pi i(n+m)x} = e_{n+m}$, and $e_0 = 1$, so this family stays closed under multiplication (and clearly addition and scalar multiplication by definition), so is an algebra which contains constant functions. Also, if $x_1 \neq x_2$ and $x_1, x_2 \in [0, 1)$, then if $n \neq 0$, $e_n(x_1) = e^{2\pi i n x_1} \neq e^{2\pi i n x_2} = e_n(x_2)$, so \mathcal{A} separates points. By (complex) Stone-Weierstrass, then we know \mathcal{A} is dense in $C(\mathbb{T}, \mathbb{C})$ with respect to $\|\cdot\|_\infty$. We know $C(\mathbb{T}, \mathbb{C})$ is dense in $L^2(\mathbb{T})$ (by some mollifier argument, for example) with respect to $\|\cdot\|_{L^2(\mathbb{T})}$. So,

$$f(x) = \lim_{N \rightarrow \infty} \sum_{n=-N}^N \alpha_n e_n(x),$$

with the limit taken in the sense of $L^2(\mathbb{T})$. ■

Recall that in Hilbert spaces, TFAE:

- $\{e_n\}$ are a basis, i.e. $f = \sum_{n=-\infty}^{\infty} \alpha_n e_n = \sum_{n=-\infty}^{\infty} (f, e_n) e_n$, in $L^2(\mathbb{T})$;
- if $(f, e_n) = 0$ for every n , $f \equiv 0$ (completeness);
- $\|f\|_{L^2(\mathbb{T})}^2 = \sum_{n=-\infty}^{\infty} |(f, e_n)|^2$ (Parseval's).

With this in mind, we define:

↪ **Definition 3.1** (Fourier Series): Let

$$\hat{f}(n) := (f, e_n) = \int_0^1 f(x) e^{-2\pi i n x} dx.$$

Then, the *complex Fourier series* is defined by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

Remark 3.1: A Fourier series can be defined for any periodic function, while we only do so for 1-periodic here. If f were L -periodic, we'd define

$$\hat{f}_L(n) := \frac{1}{L} \int_0^L f(x) e^{-\frac{2\pi i n x}{L}} dx,$$

with complex Fourier series $\sum_{n=-\infty}^{\infty} \hat{f}_L(n) e^{\frac{2\pi i n x}{L}}$.

Remark 3.2: We can also make Fourier series to be real-valued, with sines and cosines, of the form

$$A_0 + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{2n\pi x}{L}\right) + B_n \sin\left(\frac{2n\pi x}{L}\right) \right],$$

for some A_n, B_n also given by inner products.

What conditions do we need on f to make this series converge? In the general L^2 -theory, we just need $f \in L^2(\mathbb{T})$. By Parseval's,

$$\|f\|_{L^2(\mathbb{T})}^2 = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2.$$

So, the operator $\hat{\cdot} : L^2(\mathbb{T}) \rightarrow \ell^2(\mathbb{C})$. Note that this implies $\lim_{n \rightarrow \infty} |\hat{f}(n)|^2 = 0$, so also $\lim_{n \rightarrow \infty} |\hat{f}(n)| = 0$. This proves the following proposition:

↪ **Proposition 3.2** (Riemann-Lebesgue Lemma): If $f \in L^2(\mathbb{T})$,

$$\lim_{n \rightarrow \infty} |\hat{f}(n)| = 0.$$

Remark 3.3: This result is *very* useful, particularly for the real Fourier Series. In particular, it tells us statements such as

$$\lim_{n \rightarrow \infty} \int_0^1 f(x) \sin(2n\pi x) dx = 0,$$

with similar for the cosine term. These are so-called “oscillatory integrals”.

While the $L^2(\mathbb{T})$ -theory is very useful for Hilbert space interpretation, we are really concerned with the partial sums

$$S_N(x) = \sum_{n=-N}^N \hat{f}(n) e^{2\pi i n x},$$

and ways it might converge. We may rewrite by definition

$$\begin{aligned} S_N(x) &= \sum_{n=-N}^N \left(\int_0^1 f(y) e^{-2\pi i n y} dy \right) e^{2\pi i n x} \\ \text{(because finite sum)} \quad &= \int_0^1 f(y) \sum_{n=-N}^N e^{2\pi i n(x-y)} dy \\ \text{(* just over } [0, 1]) \quad &= (f * D_N)(x), \quad D_N(x) := \sum_{n=-N}^N e^{2\pi i n x}. \end{aligned}$$

So in short,

$$S_N(x) = (f * D_N)(x),$$

where $D_N(x)$ is called the *Dirichlet kernel*. Let's look at some of its properties.

$$D_N(x) = 1 + \sum_{n=1}^N [e^{2\pi i n x} + e^{-2\pi i n x}],$$

so

$$\int_0^1 D_N(x) dx = \int_0^1 1 dx + \underbrace{\sum_{n=1}^N \int_0^1 (\text{some periodic functions})}_{=0} = 1,$$

by periodicity. However, $D_N(x)$ is not actually a good kernel; one can show that $\int_0^1 |D_N(x)| dx \geq C \log N$ as $N \rightarrow \infty$.

Note too that

$$\begin{aligned} D_N(x) &= \sum_{n=-N}^N e^{2\pi i n x} \\ &= \sum_{n=0}^{2N} e^{2\pi i (n-N)x} \\ &= e^{-2\pi i N x} \sum_{n=0}^{2N} (e^{2\pi i x})^n \\ &= e^{-2\pi i N x} \left(\frac{1 - e^{2\pi i (2N+1)x}}{1 - e^{2\pi i x}} \right) \quad (\text{geometric series}) \\ &= \frac{e^{-2\pi i N x} - e^{2\pi i (N+1)x}}{1 - e^{2\pi i x}} \cdot \frac{e^{-2\pi i \frac{x}{2}}}{e^{-2\pi i \frac{x}{2}}} \\ &= \frac{e^{-2\pi i (N+\frac{1}{2})x} - e^{2\pi i (N+\frac{1}{2})x}}{e^{-2\pi i \frac{x}{2}} - e^{2\pi i \frac{x}{2}}} \\ &= \frac{\sin(2\pi (N + \frac{1}{2})x)}{\sin(2\pi \frac{x}{2})}. \end{aligned}$$

This form leads nicely to the following results.

↪ **Theorem 3.2** (Pointwise Convergence): Let $f \in L^2(\mathbb{T})$ and suppose f is Lipschitz continuous at x_0 . Then,

$$S_N(x_0) \rightarrow f(x_0).$$

PROOF. Left as an exercise. ■

↪ **Theorem 3.3** (Uniform convergence): If $f \in C^2(\mathbb{T})$, then $S_N(x) \rightarrow f(x)$ uniformly on \mathbb{T} .

Remark 3.4: In fact, we see that $\hat{f}(n) = \int_0^1 f(x)e^{-2\pi i n x} dx$ is well-defined whenever $f \in L^1(\mathbb{T})$. So, we can view

$$\hat{\cdot} : L^1(\mathbb{T}) \rightarrow \ell^\infty(\mathbb{C}).$$

Remark 3.5: All the prior results can be extended to $f \in L^1(\mathbb{T})$, via density.

§3.2 Introduction to the Fourier Transform

↪ **Definition 3.2** (Fourier Transform): Let $f : \mathbb{R} \rightarrow \mathbb{C}$. Then, for any $\zeta \in \mathbb{R}$, define

$$\hat{f}(\zeta) := \int_{\mathbb{R}} f(x)e^{-2\pi i \zeta x} dx.$$

Remark 3.6: If $f \in L^1(\mathbb{R})$,

$$|\hat{f}(\zeta)| \leq \int_{-\infty}^{\infty} |f(x)| \underbrace{|e^{-2\pi i \zeta x}|}_{=1} dx = \|f\|_{L^1(\mathbb{R})}$$

so in particular, $\hat{f} \in L^\infty(\mathbb{R})$. Moreover,

$$\begin{aligned} |\hat{f}(\zeta + h) - \hat{f}(\zeta)| &= \left| \int_{\mathbb{R}} f(x)e^{-2\pi i(\zeta+h)x} - f(x)e^{-2\pi i \zeta x} dx \right| \\ &= \left| \int_{\mathbb{R}} f(x)e^{-2\pi i \zeta x} (e^{-2\pi i h x} - 1) dx \right| \\ &\leq \int_{\mathbb{R}} |f(x)| |e^{-2\pi i h x} - 1| dx. \end{aligned}$$

We have that

$$\lim_{h \rightarrow 0} |e^{-2\pi i h x} - 1| = 0$$

for a.e. $x \in \mathbb{R}$, and

$$\int_{\mathbb{R}} |f(x)| |e^{-2\pi i h x} - 1| dx \leq 2 \int_{\mathbb{R}} |f(x)| dx = 2\|f\|_{L^1(\mathbb{R})},$$

so we can apply dominated convergence theorem to find

$$\lim_{h \rightarrow 0} |\hat{f}(\zeta + h) - \hat{f}(\zeta)| \leq \int_{\mathbb{R}} |f(x)| \underbrace{\lim_{h \rightarrow 0} |e^{-2\pi i h x} - 1|}_{=0} dx = 0,$$

so $\hat{f} \in C(\mathbb{R})$.

↪ **Proposition 3.3** (Properties of the Fourier Transform): Let $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Then,

a. $\widehat{\tau_y f}(\zeta) = e^{-2\pi i \zeta y} \hat{f}(\zeta)$, and $\tau_\eta \widehat{f}(\zeta) = e^{2\pi i \eta(\cdot)} \widehat{f(\cdot)}(\zeta)$;

b. $\widehat{f * g} = \hat{f} \cdot \hat{g}$;

c. $\int_{\mathbb{R}} f(x) \hat{g}(x) dx = \int_{\mathbb{R}} \hat{f}(x) g(x) dx$.

PROOF. a. A change of variables gives

$$\widehat{\tau_y f}(\zeta) = \int_{\mathbb{R}} f(x - y) e^{-2\pi i \zeta x} dx = e^{-2\pi i \zeta y} \int_{\mathbb{R}} f(x) e^{-2\pi i \zeta x} dx = e^{-2\pi i \zeta y} \hat{f}(\zeta).$$

Similarly,

$$\tau_\eta \hat{f}(\zeta) = \int_{\mathbb{R}} f(x) e^{-2\pi i (\zeta - \eta)x} dx = \int_{\mathbb{R}} f(x) e^{2\pi i \eta x} \cdot e^{-2\pi i \zeta x} dx = e^{2\pi i \eta(\cdot)} \widehat{f(\cdot)}(\zeta).$$

b. First, by Young's inequality $f * g \in L^1(\mathbb{R})$ so this makes sense. Moreover, since $f, g \in L^1(\mathbb{R})$, everything we need to be finite is finite, so we can apply Fubini's theorem to find

$$\begin{aligned} \widehat{f * g}(\zeta) &= \int \left(\int f(x - y) g(y) dy \right) e^{-2\pi i \zeta x} dx \\ &= \int \left(\int f(x - y) e^{-2\pi i \zeta x} dx \right) g(y) dy \\ &= \int \left(\int f(x - y) e^{-2\pi i \zeta (x - y)} dx \right) e^{-2\pi i \zeta y} g(y) dy \\ &= \left(\int f(x) e^{-2\pi i \zeta x} dx \right) \left(\int g(y) e^{-2\pi i \zeta y} dy \right) = \hat{f}(\zeta) \cdot \hat{g}(\zeta), \end{aligned}$$

where we “multiply by 1” in the second to last line to change variables in the appropriate way.

c. We can apply Fubini's again,

$$\begin{aligned} \int f(x) \hat{g}(x) dx &= \int f(x) \left(\int g(y) e^{-2\pi i x y} dy \right) dx \\ &= \int g(y) \left(\int f(x) e^{-2\pi i x y} dx \right) dy \\ &= \int g(y) \hat{f}(y) dy. \end{aligned}$$

■

↪ **Lemma 3.1:** Let $f(x) = e^{-\pi a x^2}$ for $a > 0$. Then,

$$\hat{f}(\zeta) = \frac{1}{\sqrt{a}} e^{-\pi \frac{\zeta^2}{a}}.$$

PROOF. First, note that

$$\begin{aligned}\widehat{\frac{d}{dx}f(\zeta)} &= \int_{-\infty}^{\infty} f'(x)e^{-2\pi i\zeta x} dx \\ &= f(x)e^{-2\pi i\zeta x} \Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-2\pi i\zeta)e^{-2\pi i\zeta x} dx.\end{aligned}$$

Specifying $f(x) = e^{-\pi ax^2}$, this becomes

$$\begin{aligned}&= e^{-\pi ax^2} \cdot e^{-2\pi i\zeta x} \Big|_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-\pi ax^2}(-2\pi i\zeta)e^{-2\pi i\zeta x} dx \\ &= 2\pi i\zeta \cdot \int_{-\infty}^{\infty} e^{-\pi ax^2} e^{-2\pi i\zeta x} dx = 2\pi i\zeta \cdot \hat{f}(\zeta).\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{d}{d\zeta}\hat{f}(\zeta) &= \frac{d}{d\zeta} \left(\int_{-\infty}^{\infty} f(x)e^{-2\pi i\zeta x} dx \right) \\ &= \int_{-\infty}^{\infty} f(x)(-2\pi ix)e^{-2\pi i\zeta x} dx,\end{aligned}$$

assuming finiteness; indeed,

$$\begin{aligned}\left| \int_{-\infty}^{\infty} e^{-\pi ax^2}(-2\pi ix)e^{-2\pi i\zeta x} dx \right| &\leq C \int_{-\infty}^{\infty} |x|e^{-\pi ax^2} dx \\ &= 2C \int_0^{\infty} xe^{-\pi ax^2} dx = Ce^{-\pi ax^2} \Big|_0^{\infty} < \infty,\end{aligned}$$

so our differentiation was valid. Thus, combining these two,

$$\begin{aligned}\frac{d}{d\zeta}\hat{f}(\zeta) &= \int_{-\infty}^{\infty} -2\pi ix f(x)e^{-2\pi i\zeta x} dx \\ &= \int_{-\infty}^{\infty} i(-2\pi xe^{-\pi ax^2})e^{-2\pi i\zeta x} dx \\ &= \int_{-\infty}^{\infty} \frac{i}{a} f'(x)e^{-2\pi i\zeta x} dx \\ &= \frac{i}{a} 2\pi i\zeta \hat{f}(\zeta) \\ &\Rightarrow \frac{d}{d\zeta}\hat{f}(\zeta) = -\frac{2\pi}{a}\zeta \hat{f}(\zeta).\end{aligned}$$

Thus,

$$\frac{d}{d\zeta} \left(e^{\frac{\pi\zeta^2}{a}} \hat{f}(\zeta) \right) = e^{(\pi\zeta^2)/a} \left(-\frac{2\pi}{a}\zeta \hat{f}(\zeta) \right) + \frac{2\pi\zeta}{a} e^{(\pi\zeta^2)/a} \hat{f}(\zeta) = 0,$$

and thus $e^{\frac{\pi\zeta^2}{a}} \hat{f}(\zeta)$ is constant in ζ so $e^{\frac{\pi\zeta^2}{a}} \hat{f}(\zeta) = \hat{f}(0) = \int_{-\infty}^{\infty} e^{-\pi ax^2} dx = \frac{1}{\sqrt{a}}$. Thus, $\hat{f}(\zeta) = \frac{1}{\sqrt{a}} e^{-\frac{\pi\zeta^2}{a}}$. ■

With this, we are ready to define the inverse Fourier transform;

↪ **Definition 3.3** (Inverse Fourier Transform): If $f \in L^1(\mathbb{R})$, then

$$\check{f}(x) := \int_{\mathbb{R}} f(\zeta) e^{2\pi i \zeta x} d\zeta = \widehat{f(-\cdot)}(x).$$

Remark 3.7: By similar computations to before, $f \in L^1(\mathbb{R})$ implies $\check{f} \in L^\infty(\mathbb{R}) \cap C(\mathbb{R})$.

Remark 3.8: One would hope $\check{\check{f}} = f$. However, if we check, naively,

$$\check{\check{f}}(x) = \int \left(\int f(y) e^{-2\pi i \zeta y} dy \right) e^{2\pi i \zeta x} d\zeta;$$

however the integral may not be finite in general, i.e. we cannot switch the integrals for free. We must be more careful, in short.

↪ **Theorem 3.4** (Fourier Inversion): If $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then f agrees almost everywhere with some $f_0 \in C(\mathbb{R})$, and $\check{\check{f}} = \hat{f} = f_0$.

PROOF. Let $\varepsilon > 0$ and $x \in \mathbb{R}$. Let $\varphi(\zeta) := e^{2\pi i x \zeta} e^{-\pi \varepsilon \zeta^2}$. Then,

$$\begin{aligned} \hat{\varphi}(y) &= \int \varphi(\zeta) e^{-2\pi i y \zeta} d\zeta \\ &= \int e^{2\pi i x \zeta} e^{-\pi \varepsilon \zeta^2} e^{-2\pi i y \zeta} d\zeta \\ &= e^{2\pi i x(\cdot)} \widehat{e^{-\pi \varepsilon(\cdot)^2}}(y) \\ &= \tau_x \widehat{e^{-\pi \varepsilon(\cdot)^2}}(y) \\ &= \tau_x \left(\frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi y^2}{\varepsilon}} \right) \\ &= \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi}{\varepsilon}(y-x)^2}. \end{aligned}$$

Since $\int f \hat{\varphi} dy = \int \hat{f} \varphi dy$, we find

$$\int f(y) \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi}{\varepsilon}(x-y)^2} dy = \int \hat{f}(y) \varphi(y) dy.$$

Let $K_\varepsilon(y) := \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi}{\varepsilon} y^2}$. Recall that this is the good kernel that we used in the proof of the Weierstrass Approximation Theorem. In particular, the formula above can be written

$$(f * K_\varepsilon)(x) = \int \hat{f}(y) e^{2\pi i x y} e^{-\pi \varepsilon y^2} dy. \quad \textcircled{*}$$

Recall that if f is continuous at x and compactly-supported, then $\lim_{\varepsilon \rightarrow 0} |(f * K_\varepsilon)(x) - f(x)| = 0$. This implies that for every $f \in L^1(\mathbb{R})$ $\lim_{\varepsilon \rightarrow 0} \|(f * K_\varepsilon) - f\|_1 = 0$, by an approximation argument by $C_c(\mathbb{R})$. So, taking $\varepsilon \rightarrow 0$ in $\textcircled{*}$, $f(x) \stackrel{\text{a.e.}}{=} \lim_{\varepsilon \rightarrow 0} \int \hat{f}(y) e^{2\pi i x y} e^{-\pi \varepsilon y^2} dy$. $\hat{f} \in L^1(\mathbb{R})$, so by DCT we can pass the limit inside, so

$$f(x) \stackrel{\text{a.e.}}{=} \int \hat{f}(y) e^{2\pi i x y} dy = \check{\hat{f}}(x).$$

This equality in $L^1(\mathbb{R})$ thus gives $\check{\hat{f}} = f$ a.e.. A similar proof follows for showing $\hat{\hat{f}} = f$ a.e. by replacing $e^{2\pi i x}$ with $e^{-2\pi i x}$ everywhere it appears. Since $\hat{f}, \check{\hat{f}}$ are continuous by our remarks earlier, it follows that f is equal to a continuous function almost everywhere. ■

So far, all we've worked with is $f \in L^1(\mathbb{R})$, which results in $\hat{f} \in L^\infty(\mathbb{R})$. Really, we'd like to extend the Fourier transform to act on $L^2(\mathbb{R})$, since this is a nice Hilbert space. To do so, we need the following:

→ Theorem 3.5 (Plancherel's Theorem): Let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then $\hat{f} \in L^2(\mathbb{R})$ and $\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}$.

Remark 3.9: One can view Plancherel's Theorem as a type of continuous analog of Parseval's identity for Fourier Series.

PROOF. Let $f(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, and put $g(x) := \overline{f(-x)}$, noting that then $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ as well. Put

$$w(x) := (f * g)(x).$$

By Young's,

$$\|w\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^1(\mathbb{R})} < \infty$$

so $w \in L^1(\mathbb{R})$.

We claim w continuous at 0. For h sufficiently small, we find

$$\begin{aligned} |w(h) - w(0)| &= \left| \int_{\mathbb{R}} f(h-y)g(y) dy - \int_{\mathbb{R}} f(-y)g(y) dy \right| \\ &= \left| \int_{\mathbb{R}} (f(h-y) - f(-y))g(y) dy \right| \\ &\leq \|f(h-\cdot) - f(-\cdot)\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})} \\ &\leq \|\tau_h f - f\|_{L^2(\mathbb{R})} \|g\|_{L^2(\mathbb{R})}. \end{aligned}$$

Let $\tilde{f} \in C_c(\mathbb{R})$ such that $\|f - \tilde{f}\|_{L^2(\mathbb{R})} < \eta$ for some small $\eta > 0$. Then we further bound

$$\begin{aligned}
\|\tau_h f - f\|_{L^2(\mathbb{R})} &\leq \|\tau_h f - \tau_h \tilde{f}\|_{L^2(\mathbb{R})} + \|\tau_h \tilde{f} - \tilde{f}\|_{L^2(\mathbb{R})} + \|\tilde{f} - f\|_{L^2(\mathbb{R})} \\
(\text{since norm translation invariant}) \quad &= 2\|\tilde{f} - f\|_{L^2(\mathbb{R})} + \|\tau_h \tilde{f} - \tilde{f}\|_{L^2(\mathbb{R})} \\
&\leq 2\eta + \|\tau_h \tilde{f} - \tilde{f}\|_{L^2(\mathbb{R})}.
\end{aligned}$$

Now, $\tilde{f} \in C_c(\mathbb{R})$ and thus is uniformly continuous hence $|\tau_h \tilde{f}(x) - \tilde{f}(x)| \rightarrow 0$ uniformly on \mathbb{R} hence $\|\tau_h \tilde{f} - \tilde{f}\|_{L^2(\mathbb{R})} \rightarrow 0$ as well, as $h \rightarrow 0$. Finally, since $\|g\|_{L^2(\mathbb{R})}$ finite, we conclude indeed w continuous at 0.

Next, notice that $\hat{w} = \hat{f} \cdot \hat{g}$, and

$$\begin{aligned}
\hat{g}(\zeta) &= \widehat{f(-\cdot)}(\zeta) = \int_{\mathbb{R}} \overline{f(-x)} e^{-2\pi i x \zeta} dx \\
&= \int_{\mathbb{R}} \overline{f(-x)} e^{2\pi i x \zeta} dx \\
&= \overline{\int_{\mathbb{R}} f(-x) e^{2\pi i x \zeta} dx} \\
&= \overline{\int_{\mathbb{R}} f(x) e^{-2\pi i x \zeta} dx} \\
&= \overline{\hat{f}(\zeta)},
\end{aligned}$$

so

$$\hat{w} = \hat{f} \cdot \overline{\hat{f}} = |\hat{f}|^2 \geq 0.$$

Recall our good kernel from the Weierstrass Approximation Theorem, $K_\varepsilon(y) = \frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi}{\varepsilon} y^2} = \widehat{e^{-\pi \varepsilon (\cdot)^2}}$. So,

$$\begin{aligned}
\int \hat{w}(y) e^{-\pi \varepsilon y^2} dy &= \int w(y) \frac{1}{\sqrt{\varepsilon}} e^{-(\pi y^2)/\varepsilon} dy \\
&= \int w(y) K_\varepsilon(y) dy \\
(\text{by symmetry}) \quad &= \int w(y) K_\varepsilon(-y) dy \\
&= (w * K_\varepsilon)(0).
\end{aligned}$$

On the LHS, $\hat{w} \geq 0$ so $\hat{w}(y) e^{-\pi \varepsilon y^2} \nearrow_{\varepsilon \rightarrow 0^+} \hat{w}(y)$ so by monotone convergence, $\int \hat{w} e^{-\pi \varepsilon y^2} dy \xrightarrow{\varepsilon \rightarrow 0} \int \hat{w}(y) dy$. On the other hand, we claim $(w * K_\varepsilon)(0) \xrightarrow{\varepsilon \rightarrow 0} w(0)$ (this isn't immediate from the fact that K_ε is a good kernel because we don't know a priori that w (essentially) bounded). Supposing this claim holds, this implies $\int \hat{w}(y) dy = w(0)$, hence

$$\begin{aligned}
\int \hat{w}(y) \, dy &= \int |\hat{f}(y)|^2 \, dy = w(0) \\
&= (f * g)(0) \\
&= \int f(y) \overline{f(0 - (-y))} \, dy \\
&= \int |f(y)|^2 \, dy,
\end{aligned}$$

which precisely means $\|\hat{f}\|_{L^2(\mathbb{R})} = \|f\|_{L^2(\mathbb{R})}$.

To prove the claim of $(K_\varepsilon * w)(0) \rightarrow w(0)$, let $\eta > 0$. Since w continuous at 0, there is a $\delta > 0$ such that $|y| < \delta \Rightarrow |w(y) - w(0)| < \eta$. Then,

$$\begin{aligned}
\left| \int w(0 - y) K_\varepsilon(y) \, dy - w(0) \right| &= \left| \int (w(-y) - w(0)) K_\varepsilon(y) \, dy \right| \\
&\leq \eta \int_{|y| < \delta} K_\varepsilon(y) \, dy + \int_{|y| > \delta} |w(0)| K_\varepsilon(y) \, dy + \int_{|y| > \delta} |w(-y)| K_\varepsilon(y) \, dy \\
&\leq \eta \cdot 1 + \underbrace{|w(0)|}_{\substack{w \text{ cnts at } 0 \\ \text{so this finite}}} \cdot \underbrace{\int_{|y| > \delta} K_\varepsilon(y) \, dy}_{\substack{\rightarrow 0 \text{ since good kernel} \\ \varepsilon \rightarrow 0}} + \int_{|y| > \delta} |w(-y)| K_\varepsilon(y) \, dy.
\end{aligned}$$

It remains to show the last term $\rightarrow 0$. We have

$$\begin{aligned}
\int_{|y| > \delta} |w(-y)| K_\varepsilon(y) \, dy &\leq \int_{|y| > \delta} |w(-y)| \frac{1}{\sqrt{\varepsilon}} e^{-\pi \frac{\delta^2}{\varepsilon}} \, dy \\
&\leq \underbrace{\frac{1}{\sqrt{\varepsilon}} e^{-\pi \frac{\delta^2}{\varepsilon}}}_{\rightarrow 0 \text{ as } \varepsilon \rightarrow 0} \cdot \|w\|_{L^1(\mathbb{R})} \rightarrow 0.
\end{aligned}$$

This completes the proof. ■

With these, we can extend the definition of \hat{f} to $f \in L^2(\mathbb{R})$.

Let $f \in L^2(\mathbb{R})$. Then, there are $\{f_k\} \subseteq C_c^\infty(\mathbb{R})$ such that $f_k \rightarrow f$ in $L^2(\mathbb{R})$. Since $\{f_k\} \subseteq C_c^\infty(\mathbb{R})$, $f_k \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. So, by Plancherel's,

$$\|f_j - f_k\|_{L^2(\mathbb{R})} = \|\widehat{f_j - f_k}\|_{L^2(\mathbb{R})} = \|\hat{f}_j - \hat{f}_k\|_{L^2(\mathbb{R})}.$$

So in particular, $\{\hat{f}_k\}$ also Cauchy in $L^2(\mathbb{R})$ so by completeness converges. Thus, we simply define the Fourier transform of f as the limit of these, namely,

↪ **Definition 3.4** (Fourier Transform on $L^2(\mathbb{R})$): Let $f \in L^2(\mathbb{R})$ and $\{f_k\} \subseteq C_c^\infty(\mathbb{R})$ such that $f_k \rightarrow f$ in $L^2(\mathbb{R})$. Then, we define the Fourier transform of f to be

$$\hat{f}(\zeta) := \lim_{j \rightarrow \infty} \hat{f}_j(\zeta),$$

with the limit taken in $L^2(\mathbb{R})$.

It's not obvious that this is well-defined a priori. Let f_k, f'_k be two sequences in $C_c^\infty(\mathbb{R})$ converging to f in $L^2(\mathbb{R})$, and suppose $\hat{f}_k \rightarrow \hat{f}, \hat{f}'_k \rightarrow \hat{f}'$ in $L^2(\mathbb{R})$. We need to show that $\hat{f} = \hat{f}'$ in $L^2(\mathbb{R})$. Since $f_k, f'_k \rightarrow f$ in $L^2(\mathbb{R})$,

$$\|f_k - f'_k\|_2 \rightarrow 0,$$

so also by Plancherel's,

$$\|\hat{f}_k - \hat{f}'_k\|_2 \rightarrow \|\hat{f} - \hat{f}'\|_2 = 0.$$

Denote by $C_0(\mathbb{R}) := \{f \in C(\mathbb{R}) \mid \lim_{|x| \rightarrow \infty} |f(x)| = 0\}$.

↪ **Proposition 3.4:**

1. If $xf(x) \in L^1(\mathbb{R})$, $\hat{f} \in C^1(\mathbb{R})$ and $\partial_\zeta \hat{f}(\zeta) = (-2\pi i \widehat{(\cdot x)})f(\cdot)(\zeta)$
2. If $f \in C^1(\mathbb{R}) \cap C_0(\mathbb{R})$ and $\partial_x f \in L^1(\mathbb{R})$, then $\widehat{\partial_x f}(\zeta) = (2\pi i \zeta) \hat{f}(\zeta)$
3. If $f \in L^1(\mathbb{R})$, then $\hat{f} \in C_0(\mathbb{R})$ ("Riemann-Lebesgue" type result)

PROOF. We prove only 3. If $f \in L^1(\mathbb{R})$, let $\{g_n\} \subseteq C^1(\mathbb{R}) \cap C_c(\mathbb{R})$ such that $g_n \rightarrow f$ in $L^1(\mathbb{R})$. Then, g'_n are compactly supported and continuous so $g'_n \in L^1(\mathbb{R})$. Thus, $\widehat{g'_n} \in L^\infty(\mathbb{R})$. By 2., $\widehat{g'_n}(\zeta) = (2\pi i \zeta) \widehat{g_n}(\zeta) \in L^\infty(\mathbb{R})$. Thus is only possible if $\widehat{g_n} \in C_0(\mathbb{R})$.

Since $\|g_n - f\|_1 \rightarrow 0$,

$$\|\widehat{g_n} - \hat{f}\|_\infty = \sup_\zeta \left| \int (g_n(x) - f(x)) e^{-2\pi i \zeta x} dx \right| \leq \|g_n - f\|_1 \rightarrow 0,$$

so $\widehat{g_n} \rightarrow \hat{f}$ in L^∞ . Finally, for any n ,

$$\lim_{|\zeta| \rightarrow \infty} |\hat{f}(\zeta)| \leq \underbrace{\lim_{|\zeta| \rightarrow \infty} |\widehat{g_n}(\zeta)|}_{=0} + \|\hat{f} - \widehat{g_n}\|_\infty.$$

Sending then $n \rightarrow \infty$, we know that $\|\hat{f} - \widehat{g_n}\|_\infty \rightarrow 0$, completing the proof. ■

Remark 3.10: Properties 1., 2. here can be extended to $f \in L^2(\mathbb{R})$ and $\partial_x f \in L^2(\mathbb{R})$, but require more delicate mollifying arguments. 3., however, does not extend.

Remark 3.11: Why is it important to extend $\hat{f}(\zeta)$ to $f \in L^2(\zeta)$? One reason is the analysis of Sobolev Spaces.

The final topic we'll cover is how we can relate Fourier Series to the Fourier Transform.

↪ **Theorem 3.6:** If $f \in L^1(\mathbb{R})$, then there exists $Pf : \mathbb{T} \rightarrow \mathbb{C}$ defined by

$$Pf(x) = \sum_{k \in \mathbb{Z}} \tau_k f(x) = \sum_{k \in \mathbb{Z}} f(x - k)$$

(that is, we tacitly claim this summation converges pointwise a.e. and in $L^1(\mathbb{T})$), and

$$\|Pf\|_{L^1(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{R})}.$$

Also, for every $k \in \mathbb{Z}$,

$$\hat{f}(k) = \widehat{Pf}(k),$$

where the first $\hat{\cdot}$ is the Fourier transform on \mathbb{R} and the second on \mathbb{T} .

PROOF. Let $Q = [-\frac{1}{2}, \frac{1}{2})$ so $\mathbb{R} = \bigsqcup_{j \in \mathbb{Z}} Q + j$. Then,

$$\begin{aligned} \int_{\mathbb{R}} |f(x)| \, dx &= \sum_{j \in \mathbb{Z}} \int_{Q+j} |f(x)| \, dx \\ &= \sum_{j \in \mathbb{Z}} \int_Q \underbrace{|f(x-j)|}_{\geq 0} \, dx \\ \text{(Tonelli's)} \quad &= \int_Q \sum_{j \in \mathbb{Z}} |f(x-j)| \, dx. \end{aligned}$$

Thus,

$$\int_Q \sum_{j \in \mathbb{Z}} \tau_j f(x) \, dx \leq \int_Q \sum_{j \in \mathbb{Z}} |f(x-j)| \, dx = \|f\|_{L^1(\mathbb{R})}.$$

So, Pf as defined above has

$$\|Pf\|_{L^1(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{R})},$$

and also Pf is finite a.e.. Hence, the sum in question defining $Pf(x)$ converges a.e..

Moreover,

$$\begin{aligned} \widehat{Pf}(k) &= \int_Q \underbrace{\sum_{j \in \mathbb{Z}} f(x-j)}_{\in L^1(Q)} e^{-2\pi i k x} \, dx \\ \text{(By Fubini)} \quad &= \sum_{j \in \mathbb{Z}} \int_Q f(x-j) e^{-2\pi i k x} \, dx \\ &= \sum_{j \in \mathbb{Z}} \int_{Q-j} f(x) \underbrace{e^{-2\pi i k(x+j)}}_{\substack{= e^{-2\pi i k x} \\ \text{since } e^{-2\pi i k j} = 1}} \, dx \\ &= \int_{\mathbb{R}} f(x) e^{-2\pi i k x} \, dx = \hat{f}(k). \end{aligned}$$

■

This series Pf is called the *periodization* of f .

↪ **Theorem 3.7** (Poisson Summation Formula): Let $f \in C(\mathbb{R})$ such that there are $C, \varepsilon > 0$ such that $|f(x)| \leq C(1 + |x|)^{-(1+\varepsilon)}$ (so namely $f \in L^1(\mathbb{R})$) and similarly $|\hat{f}(\zeta)| \leq C(1 + |\zeta|)^{-(1+\varepsilon)}$. Then,

$$\sum_{k \in \mathbb{Z}} f(x + k) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k x},$$

where both series converge absolutely and uniformly on \mathbb{T} . In particular, if $x = 0$,

$$\sum_{k \in \mathbb{Z}} f(k) = \sum_{k \in \mathbb{Z}} \hat{f}(k).$$

Remark 3.12: By the last remark, $\hat{f}(k) = \widehat{Pf}(k)$. So, this theorem says “periodized f ” = Pf = “Fourier series of Pf ”.

PROOF. Fix $x \in \mathbb{R}$ then

$$\begin{aligned} \left| \sum_{k \in \mathbb{Z}} f(x + k) \right| &\leq \sum_{k \in \mathbb{Z}} |f(x + k)| \\ &\leq \int_{\mathbb{R}} |f(x + y)| dy \\ &\leq \int_{\mathbb{R}} \frac{C}{(1 + |x + y|)^{1+\varepsilon}} dy \\ &= \int_{\mathbb{R}} \frac{C}{(1 + |y|)^{1+\varepsilon}} dy \\ &= -\frac{C}{(1 + |y|)^\varepsilon} \Big|_{y=-\infty}^{\infty} \leq C, \end{aligned}$$

hence the series absolutely converges, and since our bound is independent of x , it also converges uniformly. Since $S_N(x) := \sum_{k=-N}^N f(x + k)$ is continuous for each N and $S_N \rightarrow Pf$ uniformly, Pf itself is continuous, in $C(\mathbb{T})$ so thus also in $L^2(\mathbb{T})$. Thus, by Hilbert space theory,

$$Pf(x) = \sum_{k \in \mathbb{Z}} \widehat{Pf}(k) e^{2\pi i k x},$$

in $L^2(\mathbb{T})$. By the last result, $\widehat{Pf}(k) = \hat{f}(k)$, thus

$$Pf(x) = \sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k x}.$$

Finally, by the same computation as before, $\sum_{k \in \mathbb{Z}} \hat{f}(k) e^{2\pi i k x}$ will also converge absolutely and uniformly as well, call it $\tilde{P}f(x)$. Thus, we claim $\tilde{P}f = Pf$. Indeed, Pf is continuous, and $\tilde{P}f = \lim_{N \rightarrow \infty} \sum_{k=-N}^N \hat{f}(k) e^{2\pi i k x}$ so $\tilde{P}f$ also continuous. So,

$$\hat{S}_N(x) \xrightarrow{L^2(\mathbb{T})} Pf(x)$$

$$\hat{S}_N(x) \xrightarrow{\text{uniform}} \tilde{P}f(x),$$

and $Pf, \tilde{P}f$ are both continuous hence $Pf \equiv \tilde{P}f$. Thus, indeed $Pf = \sum \hat{f}(k)e^{2\pi i kx}$ as we aimed to show. ■