MATH249 - Complex Variables

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§1 Complex Numbers

The complex numbers are the set

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\},$$

where $i^2 = -1$. This set is readily equipped with operations of addition, subtraction, multiplication and division; given two complex numbers $a + bi, c + di$, these operations are determined by the rules

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = ac - bd + (ad + bc)i$$
$$\frac{1}{a + bi} = \frac{a - bi}{a^2 + b^2},$$

assuming in the final line that $a^2 + b^2 \neq 0$, i.e. that $a + bi \neq 0$ in $\mathbb{C}$. In particular, in the division line, we obtain the result by multiplying the top and bottom by the *conjugate* of $z := a + bi$; we denote

$$\bar{z} = a - bi,$$

noting that in particular,

$$z\bar{z} = a^2 + b^2 = |z|^2.$$

Any complex number $z = a + bi$ may be written in so-called *polar form*

$$z = r(\cos\theta + i\sin\theta), \quad r := \sqrt{a^2 + b^2} = |z|, \theta := \arg(z) = \arctan(b/a),$$

with the θ read modulo 2π. This is a useful representation for the sake of multiplication; given $z_i = r_i(\cos(\theta_i) + i\sin(\theta_i)), i = 1, 2$, we have

$$z_1 z_2 = \cdots = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

In particular,

$$|z_1 z_2| = |z_1||z_2|, \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

↪**Theorem 1.1**: $\cos(\theta) + i\sin(\theta) = \exp(i\theta)$

Proof. Taylor expand both sides. ■

In particular, this theorem gives a clear way to define the exponential of a complex number

$$e^{x+iy} = e^x e^{iy} = e^x(\cos(y) + i\sin(y)),$$

so that in particular, for any $z \in \mathbb{C}$,

$$|e^z| = e^{\mathrm{Re}(z)}, \quad \arg(e^z) = \mathrm{Im}(z).$$

§1.1 Fundamental Theorem of Algebra

↪**Theorem 1.2** (Fundamental Theorem of Algebra): If $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ is a polynomial with complex coefficients $a_0, a_1, \ldots, a_{n-1}, a_n$, then there exists a $z \in \mathbb{C}$ such that $f(z) = 0$.

Proof. *(A First Proof)* Remark that if $|z| = R \gg 1$ (much larger than zero), then we have

$$\begin{aligned} |a_n z^n| &= |a_n| R^n, \\ |a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| &\le |a_{n-1}| R^{n-1} + \cdots + |a_1| R + |a_0| \\ &\le (|a_{n-1}| + |a_{n-2}| + \cdots + |a_1| + |a_0|) R^{n-1}. \end{aligned}$$

Let $z_0 \in \mathbb{C}$ be a point for which $|f(z_0)|$ is a minimum; this must exist for $|f|$ must be very large outside of the disc of radius R centered at the origin. Namely, $|z_0| < R$. We claim z_0 a root of f. We may assume without loss of generality that $z_0 = 0$, by replacing $f(z)$ with $f(z - z_0)$. We write

$$\begin{aligned} f(z) &= a_0 + \cdots + a_k z^k + \cdots + a_n z^n, \\ &= a_0 + a_k z^k \left(1 + \frac{a_{k+1}}{a_k} z + \cdots + \frac{a_n}{a_k} z^{n-k} \right). \end{aligned}$$

where $a_k \neq 0$ the first nonzero coefficient with $k \geq 1$. If we can show $a_0 = 0$, we are done. Assume otherwise. Let

$$z := \left(-\frac{a_0}{a_k} \right)^{\frac{1}{k}} \varepsilon, \quad \varepsilon > 0.$$

With this value of z, we have

$$f(z) = a_0 - a_0 \varepsilon^k \left(1 + \underbrace{\cdots}_{= o(\varepsilon)} \right) \approx a_0 \left(1 - \varepsilon^k \right).$$

By choosing ε sufficiently small, this implies

$$|f(z)| < |a_0| = |f(0)|,$$

which contradicts the assumed minimality of $z_0 = 0$, unless of course $a_0 = f(z_0) = 0$, providing the claim. ■

Proof. (A Second Proof) We want to view $f(z)$ as a mapping $\mathbb{C} \to \mathbb{C}$. Assume $f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$. When $|z|$ large, we know

$$|f(z) - z^n| < C |z|^{n-1},$$

for some constant C independent of z. Remark that the map $\varphi : z \mapsto z^n$ maps a circle of radius R to a circle of radius R^n; in particular, if we take a point $z = Re^{i\theta}$ on the circle of radius R of angle θ with the origin, and let θ vary from 0 to 2π, one "rotation" in the pre-image world will lead to n "rotations" in the image world. Similarly, for $z \mapsto f(z)$, the image of the R-radius circle may not be a circle, but a "fudged" circle; the curve of the image will still be some periodic curve. As we let $R \to 0$, though, the image will go to the singular point a_0. Thus, at some value of R, the image of the R-radius circle would have to pass through the origin, and thus this point must be a root of $f(z)$. ■

PROOF. (A Third Proof) We use a result that we will prove later in the class, Liouville's Theorem, which states that any bounded differentiable function $f : \mathbb{C} \to \mathbb{C}$ must be constant.

Suppose $p(z)$ a polynomial with no roots in $\mathbb{C}$. Let $f(z) = \frac{1}{p(z)}$ (this is well-defined, since by assumption p has no roots); this is bounded on $\mathbb{C}$, and has derivative $\frac{\mathrm{d}}{\mathrm{d}z} f(z) = -\frac{p'(z)}{p(z)}$. By Liouville's, f must be a constant and thus p must be a constant. ■

§1.2 Analytic, Holomorphic Functions

↪**Definition 1.1** (Holomorphic/Analytic): A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *holomorphic* is it has a well-defined derivative, i.e. if the limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists and is well-defined (in the sense that it is independent of the "path" h takes to 0).

We may write $f : \mathbb{C} \to \mathbb{C}$ as

$$f(z) = f(x+iy) = u(x,y) + iv(x,y),$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$. We can calculate $f'(z)$ in two different ways.

1. Restrict h to $\mathbb{R}$:

$$\begin{aligned} f'(z) = \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{f(z+h) - f(z)}{h} &= \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \\ &= \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{v(x+h,y) - v(x,y)}{h} \\ &= \frac{\partial u}{\partial x}(x,y) + i \frac{\partial v}{\partial x}(x,y). \end{aligned}$$

2. Rescrict to h purely imaginary values:

$$\begin{aligned} f'(z) = \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{f(z+ih) - f(z)}{ih} &= \lim_{\substack{h \to 0, \\ h \in \mathbb{R}}} \frac{u(x,y+h) + iv(x,y+h) - u(x,y) - iv(x,y)}{ih} \\ &= \frac{1}{i} \frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) \\ &= \frac{\partial v}{\partial y}(x,y) - i \frac{\partial u}{\partial y}(x,y) \end{aligned}$$

These two computations must of course agree, which imply (equating real, imaginary parts)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

These are the *Cauchy-Riemann equations*. Viewing the pair $f = (u, v)$ as a mapping $\mathbb{R}^2 \to \mathbb{R}^2$, the Cauchy-Riemann equations imply that the Jacobian of f is given in the form

$$J_f(x, y) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a, b \in \mathbb{R}.$$

↪**Proposition 1.1**:

- If f, g are holomorphic and $a, b \in \mathbb{C}$, then $af + bg$ are also holomorphic, and moreover $(af + bg)' = af' + bg'$
- With $f(z) := z^n, f'(z) = nz^{n-1}$
- As a result, any polynomial on $\mathbb{C}$ is holomorphic

↪**Theorem 1.3**: If f satisfies the Cauchy-Riemann equations, then f is holomorphic.

Proof. Write $f = u + iv$ as before. Let $h = h_1 + ih_2$. Then,

$$u(x + h_1, y + h_2) = u(x, y) + h_1 \partial_x u + h_2 \partial_y u + |h| \psi_1(h), \quad \psi_1(h) \to 0 \text{ as } h \to 0,$$

with similar for v with a remainder ψ_2. Then, by Cauchy-Riemann,

$$f(z + h) = f(z) + \left(\partial_x v - i \partial_y u\right)(h_1 + ih_2) + \psi(h)|h|, \quad \psi(h) = o(|h|).$$

Dividng both sides by h and sending $h \to 0$ gives the result. ∎

§1.3 Power Series

We say a series $\sum_{n=0}^{\infty} a_n z^n$, where $a_n, z \in \mathbb{C}$, *converges* if $\lim_{N\to\infty} \sum_{n=0}^{N} a_n z^n$ exists as a complex number. We say it a converges *absolutely* if $\lim_{N\to\infty} \sum_{n=0}^{N} |a_n||z|^n$ exists.

↪**Theorem 1.4**: Given $\sum_{n=0}^{\infty} a_n z^n$, there exists a number $0 \leq R \leq \infty$ for which

1. if $|z| < R$, then $\sum a_n z^n$ converges absolutely;
2. if $|z| > R$, then $\sum a_n z^n$ does not converge.

Furthermore,

$$\frac{1}{R} = \limsup_n |a_n|^{\frac{1}{n}}.$$

Proof. Let $L = \frac{1}{R}$ and suppose $|z| < R$. There exists some $\varepsilon > 0$ such that

$$r := (L + \varepsilon)|z| < 1.$$

There exists some N such that $L + \varepsilon > |a_n|^{\frac{1}{n}}$ for all $n > N$ by definition of limsup's; thus

$$|z||a_n|^{\frac{1}{n}} < (L + \varepsilon)|z| = r < 1$$
$$\Rightarrow |z|^n |a_n| < r^n.$$

But since $r < 1$, it follows that $\sum |a_n||z|^n$ converges by comparing to the geometric series $\sum r^n$.

If $|z| > R$, there is an $\varepsilon > 0$ so that there are infinitely-many n's for which $|a_n|^{\frac{1}{n}} > \frac{1}{R} - \varepsilon$, and so

$$|a_n|^{\frac{1}{n}}|z| > r > 1$$

hence $|a_n||z|^n > r^n$, so that $\sum |a_n||z|^n$ diverges by comparison. Moreover, we have shown that $|a_n||z|^n$ does not converge to zero, which implies the series does not even converge ("normally"). ■

⊛ **Example 1.1**:
1. $\sum_{n=0}^{\infty} n! z^n$ has $R = 0$
2. $\sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z$ with $R = \infty$.
3. $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ has $R = 1$.

↪**Theorem 1.5**: A power series $f(z) = \sum_{n=0}^{n} a_n z^n$ admits a derivative on its disc of convergence, and $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$.

Proof. Write $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ as the "potential" derivative we aim to show, remarking that this seires converges and moreover has the same radius of convergence as f since $\lim n^{\frac{1}{n}} = 1$ and thus $\limsup a_n^{\frac{1}{n}} = \limsup (n a_n)^{\frac{1}{n}}$. Write

$$f(z) = S_N(z) + E_N(z), \quad S_N(z) := \sum_{n=0}^{N} a_n z^n, E_N(z) := \sum_{n=N+1}^{\infty} a_n z^n.$$

Fix $z_0 \in D_R(0)$. We show $\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) \to 0$ as $h \to 0$. We can write

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) &= \frac{S_N(z_0+h)-S_N(z_0)}{h} - g(z_0) + \frac{E_N(z_0+h)-E_N(z_0)}{h} \\ &= \left\{\frac{S_N(z_0+h)-S_N(z_0)}{h} - S'_{N(z_0)}\right\} + \{S'_N(z_0) - g(z_0)\} + \left\{\frac{E_N(z_0+h)-E_N(z_0)}{h}\right\} \\ &= (A) + (B) + (C).\end{aligned}$$

For all $\varepsilon > 0$, there exists N_1 $|(B)| < \varepsilon$ for all $N > N_1$.

There exists N_2 such that $|(C)| < \varepsilon$ for all $N > N_2$, since we have

$$(C) = \sum_{n \geq N+1} a_n \frac{(z_0+h)^n - z_0^n}{h},$$

and

$$(z_0+h)^n - z_0^n = h\left((z_0+h)^{n-1} + (z_0+h)^{n-2} z_0 + \cdots + (z_0+h)^{n-j} z_0^j + \cdots + z_0^{n-1}\right).$$

Since $|z_0 + h|, |z_0| < r < R$ for h sufficiently small, we know

$$\left|(z_0+h)^n - z_0^n\right| \leq |h| n r^{n-1},$$

so that

$$\left|\frac{(z_0+h)^n - z_0^n}{h}\right| \leq n r^{n-1}.$$

It follows that

$$|(C)| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}.$$

This is the tail of an absolutely converging series, hence as $N \to \infty$, $|(C)| \to 0$, so we have the claimed bound.

Finally, let $N := \max(N_1, N_2)$. We see that for any fixed N, $(A) \to 0$ as $h \to 0$ by the definition of the derivative, and thus we can take $h = h(N)$ sufficiently small so that $|(A)| < \varepsilon$. Combining all these bounds gives the proof. ■

↪**Corollary 1.1**: $f(z) = \sum a_n z_n$ is infinitely differentiable in its radius of convergence.

↪**Definition 1.2**: A function $f : \Omega \to \mathbb{C}$ is called *analytic* if it is equal to a power series on $D_\varepsilon(z_0)$ for all $z_0 \in \Omega$, for some $\varepsilon > 0$.

↪**Corollary 1.2**: f analytic $\Rightarrow f$ holomorphic

Remark 1.1: We'll see later that these are actually equivalent notions.

§1.4 Integration Along Curves

↪**Definition 1.3**: A parametrized curve is a function $\gamma : [0,1] \to \mathbb{C}$ where γ is differentiable with continuous derivative, with $\gamma'(t) \neq 0$ for all $t \in [0,1]$.

↪**Definition 1.4**: We'll say two parametrized curves $\gamma, \tilde{\gamma}$ are equivalent if there exists a smooth function $s : [0,1] \to [0,1]$ smooth with $s'(t) > 0$ and such that $\tilde{\gamma} = \gamma \circ s$.

We will consider curves as defined up to equivalency in this way.

↪**Definition 1.5**: If γ is a parametrized curve, define

$$\int_\gamma f(z)\, \mathrm{d}z := \int_0^1 f(\gamma(t))\gamma'(t)\, \mathrm{d}t.$$

If γ a piecewise smooth curve, i.e. γ can locally be written as $t \mapsto z(t) \in \mathbb{C}$ for $t \in [a_k, a_{k+1})$ for $k = 0, ..., n-1$ for some sequence $a_k < a_{k+1}$, then

$$\int_\gamma f(z)\, \mathrm{d}z := \sum_{k=0}^{n+1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)\, \mathrm{d}t.$$

An obvious generalization holds for integration along more general intervals.

↪**Proposition 1.2**: Path integrals are independent of choice of parametrization.

↪**Definition 1.6** (Length of a curve): Define, for γ given by $z : I \to \mathbb{C}$,

$$\mathrm{length}(\gamma) := \int_\gamma |\mathrm{d}z| = \int_I |z'(t)|\, \mathrm{d}t.$$

↪**Proposition 1.3**: Let f, g continuous and $\alpha, \beta \in \mathbb{C}$. Then we have

1. Linearity:
$$\int_\gamma (\alpha f + \beta g)\, \mathrm{d}z = \alpha \int_\gamma f\, \mathrm{d}z + \beta \int_\gamma g\, \mathrm{d}z.$$
2. $$\int_\gamma f(z)\, \mathrm{d}z = -\int_{\gamma^-} f(z)\, \mathrm{d}z,$$
where γ^- is the *reverse path* of γ.
3. $\left|\int_\gamma f(z)\, \mathrm{d}z\right| \leq \sup_{z\in\gamma} |f(z)| \operatorname{length}(\gamma)$.

↪**Definition 1.7** (Primitive): A *primitive* of a continuous function f on a domain Ω is a function F such that $F' = f$ on Ω.

↪**Proposition 1.4**: If f, continuous, has a primitive F on Ω and γ is a curve in Ω beginning at w_1 and ending at w_2, then
$$\int_\gamma f\, \mathrm{d}z = F(w_2) - F(w_1).$$

§1.5 Cauchy's Theorem

↪**Theorem 1.6** (Cauchy): If γ is a closed path contained in a region $\Omega \subset \mathbb{C}$ and its interior, and f is holomorphic in Ω, then $\int_\gamma f(z)\, \mathrm{d}z = 0$.

It will take us some building to get here. In a simple case, though, we have a positive result:

↪**Corollary 1.3**: If f has a primitive F on Ω, then Cauchy's theorem holds for f for any γ a closed path in $\operatorname{int}(\Omega)$

PROOF. Apply the last proposition; now, $F(w_2) = F(w_1)$, so we have the result. ■

With some more work, we can also establish the proof for γ some simple contour.

↪**Proposition 1.5** (Gorsut's Lemma): Let γ be a closed triangle in Ω and f a holomorphic function on Ω. Then $\int_\gamma f(z)\, \mathrm{d}z = 0$.

PROOF. I'll add it later. ■