MATH574 - Dynamical Systems

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Contents

§1 Examples of Dynamical Systems

Roughly speaking, a dynamical system is a system that evolves in time, with common examples being a differential equation, in the continuous case, or a map, in the discrete case.

⊛ **Example 1.1** (The Logistic Map):

§2 Existence-Uniqueness Theory

↪**Definition 2.1** (Lipschitz): We say a function $f : \mathbb{R}^p \to \mathbb{R}^p$ is Lipschitz on $B \subseteq \mathbb{R}^p$ if there is a constant $L > 0$ such that $\|f(x) - f(y)\| \leq L \|x - y\|$ for every $x, y \in B$. We call L a "Lipschitz" constant. It is certainly not unique in general.

We say f *globally Lipschitz* if it is Lipschitz on $B = \mathbb{R}^p$, and f *locally Lipschitz* if f Lipschitz on every bounded domain $B \subseteq \mathbb{R}^p$ (note: the L will in general depend on the domain).

↪**Theorem 2.1**: Let $f : \mathbb{R}^p \to \mathbb{R}^p$ be a locally Lipschitz function. Then, there exists a unique solution to the initial value problem $\dot{u} = f(u), u(0) = u_0$ on some interval $t \in (-T_1(u_0), T_2(0))$, where $-T_1(u_0) < 0 < T_2(u_0)$ and

- either $T_2(u_0) = +\infty$ or $\|u(t)\| \to \infty$ as $t \to T_2(u_0)$, and
- either $T_1(u_0) = \infty$ or $\|u(t)\| \to -\infty$ as $t \to -T_1(u_0)$.

Heuristically, this first condition states that either our solution exists for all (forward) time after $-T_1(u_0)$, or it blows up in finite time, with a similar interpretation for the second, going backwards.

↪**Proposition 2.1**: Let $\dot{u} = f(u)$ where f locally Lipschitz. Let $B \subseteq \mathbb{R}^p$ be a bounded subset such that initial conditions $u_0, v_0 \in B$ define solutions $u(t), v(t)$ with $u(t), v(t) \in B$ for all $t \in [0, T]$. Let L be a Lipschitz constant for f on B. Then,

$$e^{-Lt} \|u_0 - v_0\| \leq \|u(t) - v(t)\| \leq e^{Lt} \|u_0 - v_0\| \qquad \forall\, t \in [0, T].$$

This provides a bound on how quickly solutions grow, decay in B.

↪**Corollary 2.1**: Let f locally Lipschitz and $u_0 \neq v_0$. Then, $u(t) \neq v(t)$ for all time such that the solutions both exist.

§3 Limit Sets and the Evolution Operator

We state definitions in this section first for ODEs, but they generalize.

↪**Definition 3.1** (Evolution Operator): Given $\dot{u} = f(u)$, the *evolution operator* is the map

$$S(t) : \mathbb{R}^p \to \mathbb{R}^p, \qquad t \geq 0$$

such that $u(t) = S(t)u_0$.

Such a map also defines a *semi-group* $\{S(t) : t \geq 0\}$ under composition, namely it is closed under repeated composition and this operator is associative.

For $B \subseteq \mathbb{R}^p$ define

$$S(t)B := \bigcup_{u \in B} S(t)u = \{u(t) = S(t)u_0 : u_0 \in B\}.$$

↪**Definition 3.2** (Forward/Positive Orbit): We define the *forward orbit* of a point u_0 as

$$\Gamma^+(u_0) := \bigcup_{t \geq 0} S(t)u_0,$$

i.e. the set of all points u_0 may "visit" as time increases.

↪**Definition 3.3** (Backwards/Negative Orbit): Similarly, define a *backwards orbit* (if one exists)

$$\Gamma^-(u_0) := \{u(t) : t \leq 0\},$$

s.t. $\forall\, t \leq s \leq 0, S(-t)u(t) = u_0$ and $S(s-t)u(t) = u(s)$.

Note that a negative orbit won't be unique in general, eg in maps, periodic points may multiple preimages.

↪**Definition 3.4** (Complete Orbit): If a negative orbit for u_0 exists, define the *complete orbit* through u_0 as

$$\Gamma(u_0) := \Gamma^+(u_0) \cup \Gamma^-(u_0).$$

Notice that if $v \in \Gamma(u_0)$, then $\Gamma(v) = \Gamma(u_0)$; namely a complete orbit through v exists.

↪**Definition 3.5** (Invariance): The set B is said to be *positively invariant* if $S(t)B \subseteq B$ for all $t \geq 0$. Similarly, B is said to be *negatively invariant* if $B \subseteq S(t)B$ for all $t \geq 0$.

↪**Definition 3.6** (ω-limit sets): A point $x \in \mathbb{R}^p$ is called an *ω-limit point* of u_0 if there exists a sequence $\{t_n\}$ with $t_n \to \infty$ such that $S(t_n)u_0 \to x$ as $n \to \infty$. The set of all such points for an initial condition u_0 is denoted $\omega(u_0)$, and called the ω-limit set of u_0.

Given a bounded set B, the ω-limit set of B is defined as

$$\omega(B) := \{x \in \mathbb{R}^p : \exists\, t_n \to \infty, y_n \in B \text{ s.t. } S(t_n)y_n \to x\}.$$

Remark 3.1: In general, $\omega(B)$ is *not* the union of ω-limit sets of points in B.

↪**Theorem 3.1**: For any $u_0 \in \mathbb{R}^p$,

$$\omega(u_0) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} \{S(t)u_0\}},$$

and similarly for any bounded $B \subseteq \mathbb{R}^p$,

$$\omega(B) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B}.$$

↪**Definition 3.7** (α-limit set): A point $x \in \mathbb{R}^p$ is called an *α-limit point* for $u_0 \in \mathbb{R}^p$ if there exists a negative orbit through u_0 and a sequence $\{t_n\}$ with $t_n \to -\infty$ such that $u(t_n) \to x$. The set of all such points for u_0 is denoted $\alpha(u_0)$.

↪**Theorem 3.2**: If $\Gamma^+(u_0)$ bounded, then $\omega(u_0)$ is a non-empty, compact, invariant, connected set.

↪**Definition 3.8** (Attraction): We say a set A *attracts* B if for every $\varepsilon > 0$, there is a $t^* = t^*(\varepsilon, A, B)$ such that $S(t)B \subseteq N(A, \varepsilon)$ for every $t \geq t^*$, where $N(A, \varepsilon)$ denotes the ε-neighborhood of A.

A compact, invariant set A is called an *attractor* if it attracts an open neighborhood of itself, i.e. $\exists\, \varepsilon > 0$ such that A attracts $N(A, \varepsilon)$.

A *global attractor* is an attractor that attracts every bounded subset of $\mathbb{R}^p$.

↪**Theorem 3.3** (Continuous Gronwall Lemma): Let $z(t)$ be such that $\dot{z} \leq az + b$ for some $a \neq 0, g \in \mathbb{R}$ and $z(t) \in \mathbb{R}$. Then, $\forall\, t \geq 0$,

$$z(t) \leq e^{at}z(0) + \frac{b}{a}(e^{at} - 1).$$

↪**Theorem 3.4** (ω-limit sets as attractors): Assume $B \subseteq \mathbb{R}^p$ is a bounded, open set such that $S(t)B \subseteq \overline{B} \, \forall \, t > 0$. Then, $\omega(B) \subseteq B$, and $\omega(B)$ is an attractor, which attracts B. Furthermore,

$$\omega(B) = \bigcap_{t \geq 0} S(t)B.$$

↪**Definition 3.9** (Dissipative): A dynamical system is called *dissipative* if there exists a bounded set B such $\forall \, A$ bounded, there exists a $t^* = t^*(A) > 0$ such that $S(t)A \subseteq B \;\; \forall \, t \geq t^*$. We then call such a B an *absorbing set*.

Remark 3.2: B absorbing $\Rightarrow \omega(A) \subseteq \omega(B)$. Moreover, $\omega(B)$ attracts A for every bounded set A. I.e., $\omega(B)$ is a global attractor.

§4 Stability Theory

↪**Definition 4.1** (Stable/Unstable Manifolds): If u^* a steady state of a dynamical system, the *stable manifold* of u^* is defined as the set

$$\{u \in \mathbb{R}^p : \omega(u) = u^*\},$$

and similarly, the *unstable manifold* is defined

$$\{u \in \mathbb{R}^p : \Gamma^-(u) \, \exists \text{ and } \alpha(u) = u^*\}.$$

↪**Definition 4.2** (Lyapunov Stability): A steady state u^* is called *Lyapunov stable* if $\forall \, \varepsilon > 0$, there exists a $\delta > 0$ such that if $\|u^* - v\| < \delta$, then $\|S(t)v - u^*\| < \varepsilon$ for all time $t \geq 0$.

↪**Definition 4.3** (Quasi-Asymptotically Stable): A steady state u^* is called *Quasi-asymptotically stable* (qas) if there exists a $\delta > 0$ such that if $\|u - u^*\| < \delta$, $\lim_{t \to \infty} \|S(t)u - u^*\| = 0$.

↪**Definition 4.4** (Asymptotically Stable): A steady state u^* is called asymptotically stable if it is both Lyapunov stable and qas.

↪**Definition 4.5** (Linearization): Consider a dynamical system $\dot{u} = f(u)$, where $f(u^*) = 0$. Let $v(t) = u(t) - u^*$, then, $\dot{v} = f(u^* + v)$, and $v^* = 0$ corresponds to a fixed point. Taylor expanding $\dot{v}$, we find

$$\begin{aligned}\dot{v} &= f(u^* + v) \\ &= f(u^*) + J_f(u^*)v + \mathcal{O}\left(\|v\|^2\right) \\ &= J_f(u^*) \cdot v + \mathcal{O}\left(\|v\|^2\right),\end{aligned}$$

where $J_f(u^*)$ the Jacobian matrix of f evaluated at u^*. The linear system

$$\dot{v} = J_f(u^*)v$$

is called the *linearization* of $\dot{u} = f(u)$ at u^*.

↪**Proposition 4.1**: The general solution to the linearized system

$$\dot{v} = Jv, \qquad v(0) = v_0,$$

is

$$v(t) = e^{tJ} \cdot v_0,$$

where $e^{\cdot}$ the matrix exponential defined by the (always convergent) series

$$e^M = \sum_{j=0}^{\infty} \frac{M^j}{j!}.$$

Suppose $\dot{v} = Jv$ and J complex diagonlizable with eigenvalues $\lambda_1, ..., \lambda_n$. Then, J conjugate to the diagonal matrix Λ with diagonal entries equal to the eigenvalues, namely

$$J = P\Lambda P^{-1}.$$

It follows that

$$v(t) = Pe^{t\Lambda}P^{-1}v_0.$$

Equivalently (changing coordinates), letting $w(t) = P^{-1}v(t)$, we find

$$w(t) = e^{t\Lambda}w(0),$$

noting that now, since Λ diagonal,

$$e^{t\Lambda} = \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}.$$

↪**Definition 4.6** (Linear Stable, Unstable, Centre Manifolds): Supposing 0 a steady state and $J_f(0)$ complex diagonalizable, define respectively the *linear* stable, unstable, and centre manifolds:

$$E^s(0) := \{u \mid u \text{ spanned by eigenvectors with } \Re(\lambda) < 0\}$$
$$E^u(0) := \{u \mid u \text{ spanned by eigenvectors with } \Re(\lambda) > 0\}$$
$$E^c(0) := \{u \mid u \text{ spanned by eigenvectors with } \Re(\lambda) = 0\}.$$

Notice that if $u_0 \in E^s(0)$, then the corresponding solution with initial condition u_0, $u(t)$, converges to 0 as $t \to \infty$, with similar conditions for $u_0 \in E^u(0)$.

↪**Definition 4.7** (Hyperbolic): A steady state u^* is called *hyperbolic* if $J_f(u^*)$ has no eigenvalues with $\Re(\lambda) = 0$, i.e. $\dim(E^c(u^*)) = 0$.

↪**Theorem 4.1**: If u^* a hyperbolic steady state of $\dot{u} = f(u)$, and all of the eigenvalues of $J_f(u^*)$ have strictly negative real part, then u^* is asymptotically stable.

↪**Theorem 4.2**: If u^* a steady state and $J_f(u^*)$ has a steady state with eigenvalue having real part strictly positive real part, then u^* unstable (namely not Lyapunov stable).

Remark 4.1: These theorems describe cases in which the linearization is correct in predicting the nonlinear behaviour.

Remark 4.2: The second theorem can only guarantee non-Lyapunov stability because linearization is a local process - quasi-asymptotic stability is “more global”, and not picked up by the linearization necessarily.

↪**Theorem 4.3** (Hartman-Grobman Theorem): If f continuously differentiable and $\dot{u} = f(u)$ has a hyperbolic steady state u^*, then there exists an open ball $B(u^*, \delta) \subseteq \mathbb{R}^p$, an open set $0 \in N$ and a homeomorphism

$$H : B(u^*, \delta) \to N$$

such that while $u(t) \in B(u^*, \delta)$ a solution to $\dot{u} = f(u)$, then $v(t) = H(u(t))$ a solution of $\dot{v} = J_f(u^*)v$.

↪**Definition 4.8** (Stable, Unstable Manifold): The *stable, unstable* manifolds of a steady state u^* are defined

$$W^s(u^*) := \{u \in \mathbb{R}^p \mid S(t)u \to u^* \text{ as } t \to \infty\}$$
$$W^u(u^*) := \{u \in \mathbb{R}^p \mid \Gamma^-(u) \,\exists \text{ and } S(t)u \to u^* \text{ as } t \to -\infty\}.$$

§5 Delay Differential Equations

A delay differential equation (DDE) is, broadly speaking, an ODE that depends on the state of the system in the past. We'll focus on DDEs of the form

$$\dot{u}(t) = f(u(t), u(t-\tau)),$$

where $u \in \mathbb{R}^p, f : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^p$, and $\tau > 0$ a fixed time delay.

The "canonical" first example of a DDE is $\dot{u}(t) = u(t-\tau)$ for $t \geq 0$. Notice that for any time $t \in [0, \tau]$, then, $\dot{u}(t)$ depends on u for times that are not given by the DDE directly. In short, then, we need to supply not just an initial value to the DDE, but a whole initial data, namely $u(t) = \varphi(t)$ for $t \in [-\tau, 0]$.

Suppose for now we take $\varphi \equiv 1$, so we wish to solve the DDE with initial data

$$\begin{cases} \dot{u}(t) = u(t-\tau) & t > 0 \\ u(t) = 1 & -\tau \leq t \leq 0 \end{cases}.$$

One method of solution is called the "method of steps". Note that the initial data implies

$$\dot{u}(t) = 1 \text{ for } t \in [0, \tau],$$

hence $u(t) = t + 1$ on $[0, \tau]$. Then, for $t \in [\tau, 2\tau]$,

$$\dot{u}(t) = u\Bigg(\underbrace{t-\tau}_{\in[0,\tau]}\Bigg) = (t-\tau) + 1,$$

so $u(t) = 1 + \tau + (t-\tau)(1-\tau) + \frac{1}{2}\left(t^2 - \tau^2\right)$ for $t \in [\tau, 2\tau]$. Repeating this procedure arbitrarily results in a a piecewise solution defined on each interval of the form $[n\tau, (n+1)\tau]$ for $n \in \mathbb{N}$. This method can be applied for more general DDEs, and will, in general, result in continuous solutions, differentiable everywhere except, in general, at the endpoints $n\tau$.

Another method, specifically for linear DDEs, which more related to the ODE theory, is to derive a characteristic equation. Suppose a solution of the form $u(t) = ke^{\lambda t}$ to the DDE $\dot{u}(t) = \beta u(t-\tau)$. Plugging this into the equation gives

$$k\lambda e^{\lambda t} = \beta k e^{\lambda(t-\tau)} \Rightarrow \Delta(\lambda) := \lambda - \beta e^{-\lambda\tau} = 0.$$

Solving for λ such that $\Delta(\lambda) = 0$ is, in general, difficult. However, one notices that if $\beta > 0$,

$$\lim_{\lambda \to -\infty} \Delta(\lambda) = +\infty, \qquad \delta(0) = -\beta < 0,$$

so by the intermediate value theorem, there exists at least one solution to the characteristic equation, and moreover, $\lambda \in (0, \infty)$. Similar applies for $\beta < 0$.

§5.1 DDE Linearization

Suppose we have a DDE

$$\dot{u}(t) = f(u(t), u(t-\tau)),$$

where $u \in \mathbb{R}^d$ (so $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$), with a steady-state solution u^*. Then, the linearization of the DDE about u^* is given by

$$\dot{v}(t) = Av(t) + Bv(t-\tau), \qquad v(t) := u(t) - u^*,$$

where A, B are $d \times d$ matrices given by

$$A := \frac{\partial f}{\partial u}|_{u=u^*}, \qquad B := \frac{\partial f}{\partial v}|_{u=u^*}.$$

The characteristic equation of the linearization is given

$$\Delta(\lambda) = \lambda I_d - A - Be^{-\lambda\tau} = 0.$$

For an example, consider the Mackey-Glass Equation,

$$\dot{u}(t) = -\gamma u(t) + \frac{\beta u(t-\tau)}{1 + u(t-\tau)^n}.$$

There are two steady states given by

$$u_1 = 0, \qquad u_2 = \left(\frac{\beta}{\gamma} - 1\right)^{\frac{1}{n}},$$

the second only existing when $\frac{\gamma}{\beta} < 1$. In our earlier notations, we find

$$f(u, v) = -\gamma u + \frac{\beta v}{1 + v^n}.$$

Then, $f_u = -\gamma$ and $f_v = \beta\frac{[1+(1-n)v^n]}{(1+v^n)^2}$.

§6 Bifurcation Theory

↪**Theorem 6.1** (Implicit Function Theorem): Let $f : \mathbb{R}^p \times \mathbb{R} \to \mathbb{R}^p$ be a C^1 function of (u, μ) with $f(0,0) = 0$. If $J_f = f_u(0,0)$ is invertible, then there exists $\varepsilon > 0$ and a smooth curve $u = G(\mu)$ which is the unique solution of $f(G(\mu), \mu) = 0$ for $|\mu| < \varepsilon$ and $\|u\| < \varepsilon$.

↪**Corollary 6.1**: If (u^*, μ^*) a hyperbolic steady state of $\dot{u} = f(u, \mu)$, then for some $\varepsilon > 0$ there is a smooth curve $u = G(\mu)$ with $u^* = G(\mu^*)$ whenever $\|u - u^*\| < \varepsilon$ and $|\mu - \mu^*| < \varepsilon$, such that $G(\mu)$ a steady state of $\dot{u} = f(u, \mu)$, i.e. $f(G(\mu), \mu) = 0$.

Remark 6.1: Heuristically, this means that if J_f invertible, there can be no change in the number of steady states near u^* while μ near μ^*. Similarly, small perturbations of J_f won't change the sign of the real part of the eigenvalues of J_f, hence stability won't change in this case. Thus, to study scenarios in which changes in μ qualitatively change dynamics, we need to study non-hyperbolic steady states. We call such a scenario a "bifurcation".

§6.1 Canonical 1-Dimensional Bifurcations

Suppose

$$\dot{u} = f(u, \mu), \qquad f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

has a fixed point at $(u, \mu) = (0, 0)$ (if the fixed point is at a different point, we may simple change coordinates to move it to the origin). The following table outlines the most common bifurcation types, and conditions for the to occur, in the one-dimensional case.

In the "Conditions" column, all partial derivatives are evaluated at $(0, 0)$. These conditions arise naturally from a Taylor expansion of f about the steady state, and considering different combinations of quantities being zero or nonzero.

Name	Normal Form	Conditions*	Description	Graphs
Saddle Node	$\dot{u} = \mu - u^2$	$f = f_u = 0, f_\mu \neq 0$	Single s.s. branches into 2	Figure 1
Transcritical	$\dot{u} = \mu u - u^2$	$f = f_u = f_\mu = 0, f_{uu} \neq 0, f_{u\mu}^2 > f_{\mu\mu} \cdot f_{uu}$	2 steady states pass through each other and change stability	Figure 2
Supercritical Pitchfork	$\dot{u} = \mu u - u^3$	$f = f_u = f_\mu = f_{uu} = 0, f_{uuu} \neq 0, f_{\mu u} \neq 0$	Single stable fixed point becomes unstable and two new stable fixed points are born surrounding it	Figure 3
Subcritical Pitchfork	$\dot{u} = -\mu u + u^3$	As above	Same as above, interchanging stable and unstable	

Remark 6.2: * The first two conditions, $f = f_u = 0$, which appear in all the cases, are required for a bifurcation ($f = 0$ gives a steady state, $f_u = 0$ means the implicit function theorem doesn't apply). Then, $f_\mu \neq 0$ implies a saddle-node, so the requirement $f_\mu = 0$ in the other cases just rule out not being a saddle-node. The other conditions from there are just technical, and arise from the Taylor expansion naturally.

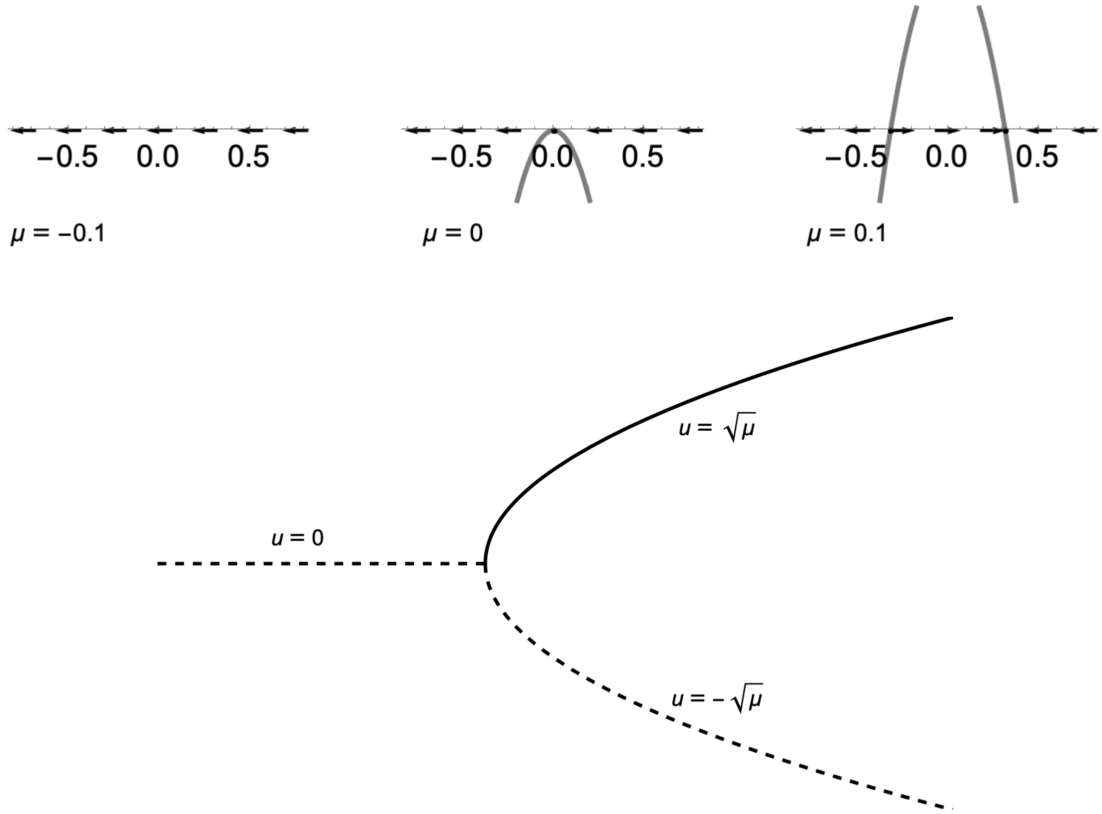


Figure 1: Vector fields for a saddle node bifurcation (above) and corresponding bifurcation diagram (below).

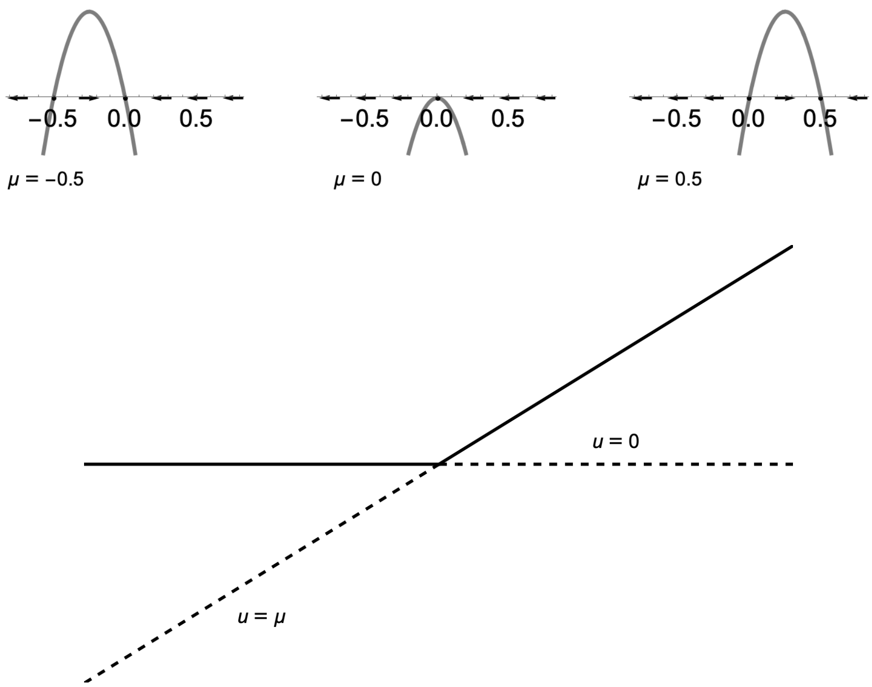


Figure 2: Vector fields for a transcritical bifurcation (above) and corresponding bifurcation diagram (below).

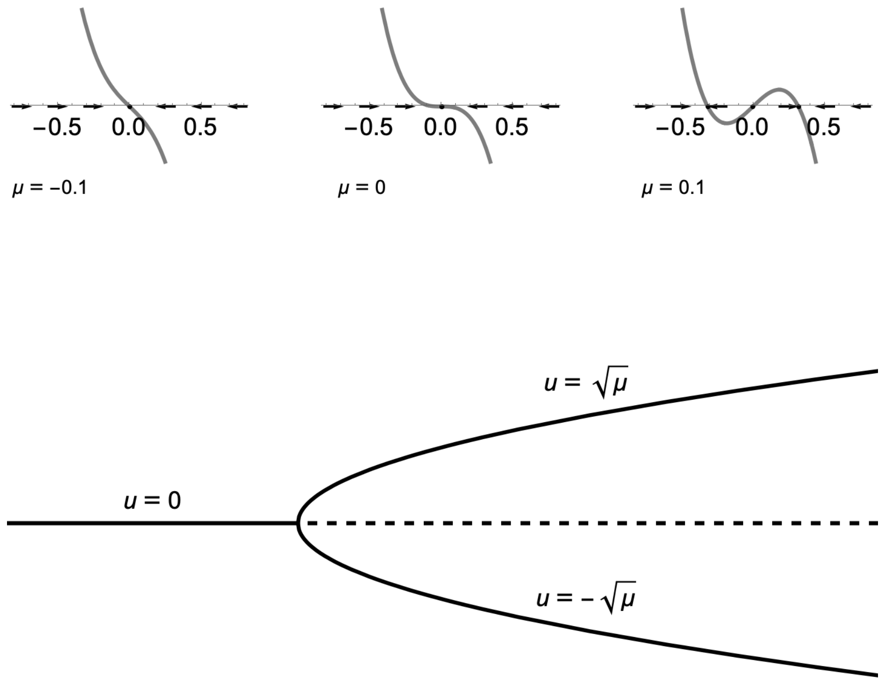


Figure 3: Vector fields for a (supercritical) pitchfork bifurcation (above) and corresponding bifurcation diagram (below).

§6.2 Bifurcations In $\mathbb{R}^p$

In higher dimensions than 1, we have slightly more complex behaviour. From the Implicit Function Theorem, we know that if the Jacobian J_f remains invertible as a parameter is varied, a steady state $u^*(\mu)$ will vary continuously with μ. So, the stability of the steady state may change with μ, but the number (locally) of steady states stays the same. For instance, this can happen in $\mathbb{R}^2$ if a complex conjugate pair of eigenvalues has real part changing sign. In this case, the steady state not hyperbolic, yet J_f remains invertible (as long as the imaginary part of the eigenvalues remain nonzero.) This is called a "Hopf bifurcation", which we'll see more of later. Otherwise, bifurcations in $\mathbb{R}^2$ occur when a single real eigenvalue changes sign. We'll deal, first off, with bifurcations that involve a single eigenvalue crossing 0 (changing sign) at a given time, which generally occurs with one-parameter systems. More generally if there are > 1 parameters in a dynamical system, it is possible to make k eigenvalues simultaneously zero, in which case we have a so-called "co-dimension k" bifurcation. We touch on these later.

↪**Theorem 6.2** (Center Manifold Theorem): Consider $\dot{u} = f(u)$ where $f \in C^r(\mathbb{R}^p, \mathbb{R}^p)$ and $f(0) = 0$. We classify the eigenvalues λ of the Jacobian of f at 0 in the following:

$$\sigma_u := \{\lambda \mid \mathrm{Re}(\lambda) > 0\}$$
$$\sigma_s := \{\lambda \mid \mathrm{Re}(\lambda) < 0\}$$
$$\sigma_c := \{\lambda \mid \mathrm{Re}(\lambda) = 0\}.$$

Denote E^u, E^s, E^c the corresponding subspaces of $\mathbb{R}^p$ (namely the spaces spanned by the eigenvectors corresponding to eigenvalues in $\sigma_u, \sigma_s, \sigma_c$ respectively). Then, there exist C^r-smooth stable, unstable manifolds W^s, W^u tangential to E^s, E^u at 0, and a C^{r-1}-smooth manifold W^c tangential to E^c at 0, with the propertie that all of these manifolds are invariant for the dynamical system.

Remark 6.3: In this theorem, W^s, W^u and W^c are not the same as those discussed before, defined using ω-limit sets; now we require both $\mathrm{Re}(\lambda) \neq 0$ *and* stability/instability criteria.

We can often approximate the manifolds in the theorem by assuming that they can be written as curves that are functions of one variable, then applying an appropriate series expansion to determine coefficients. Globally, this may not work, but locally can give a good picture of the nonlinear manifold.

⊛ **Example 6.1**: In $\mathbb{R}^2$, let

$$\dot{x} = xy, \dot{y} = -y - x^2.$$

This system has a steady state at $(0, 0)$, with

$$J_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},$$

so there are two eigenvalues, $0, -1$. Moreover, this gives $E^s = \mathrm{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $E^c = \mathrm{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

If $x(0) = 0$, $\dot{x} = 0$ so $x = 0$ for all time, hence $W^s(0,0) = E^s$ in this case.

For the nonlinear center manifold, W^c, suppose that locally, W^c is the graph of a smooth function of x, $y = h(x)$, i.e.

$$W^c = \{(x, h(x)) \mid x \in \mathbb{R}\}.$$

To compute h, suppose

$$y = h(x) = \sum_{j=0}^{\infty} c_j x^j,$$

with the coefficients c_j to be determined. By assumption, the dynamics are invariant on $h(x)$, so on the one hand

$$\dot{y} = h'(x)\dot{x} = h'(x)(xy) = xh'(x)h(x)$$

while also

$$\dot{y} = -y - x^2 = -h(x) - x^2,$$

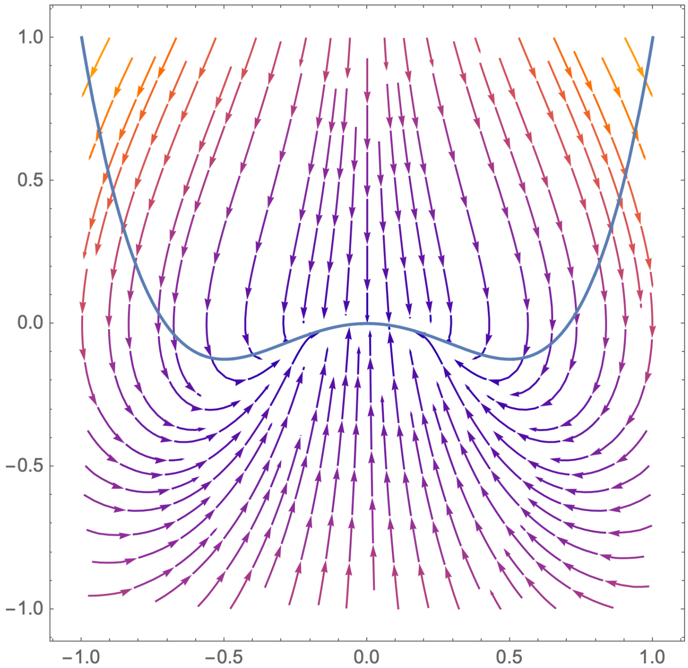
so setting these equal, we find the relation

$$xh'(x)h(x) = -h(x) - x^2.$$

Equating like terms, we find

$$c_0 = c_1 = 0, \qquad c_2 = -1, c_3 = 0, c_4 = -2, c_5 = 0,$$

etc. (note that we could have found the first two sooner; we know the curve must pass through the origin hence $c_0 = 0$, and we know it must be tangential to E^c, the x-axis, so its first derivative c_1 must also be zero). Plotting the first few terms of these curve against the actual vector field, we find:



Locally, its clear this curve is invariant under the dynamics of the system, and as we move further the approximation fails. This is because, away from the origin, the assumption that the unstable manifold could be represented as a curve parametrized in one variable fails.

⊛ **Example 6.2**: More generally, let

$$\dot{x} = x(\mu + y), \qquad \dot{y} = -y - x^2,$$

for μ near 0. Repeat the analysis of the system in the previous example (which is just this equation with $\mu = 0$). (The algebra is a little more difficult, but doable.)

§6.3 Hopf Bifurcations

The Hopf Bifurcation is most readily described by example.

⊛ **Example 6.3**: Consider the system

$$\dot{x} = \mu x - \omega y - ax(x^2 + y^2), \qquad \dot{y} = \omega x + \mu y - ay(x^2 + y^2),$$

where ω, μ, a are real parameters.

In polar coordinates, this system becomes

$$\dot{r} = \mu r - ar^3, \qquad \dot{\theta} = \omega,$$

from which we see that there is a unique fixed point at the origin. Here, the Jacobian (in Cartesian coordinates) is

$$J(0,0) = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix},$$

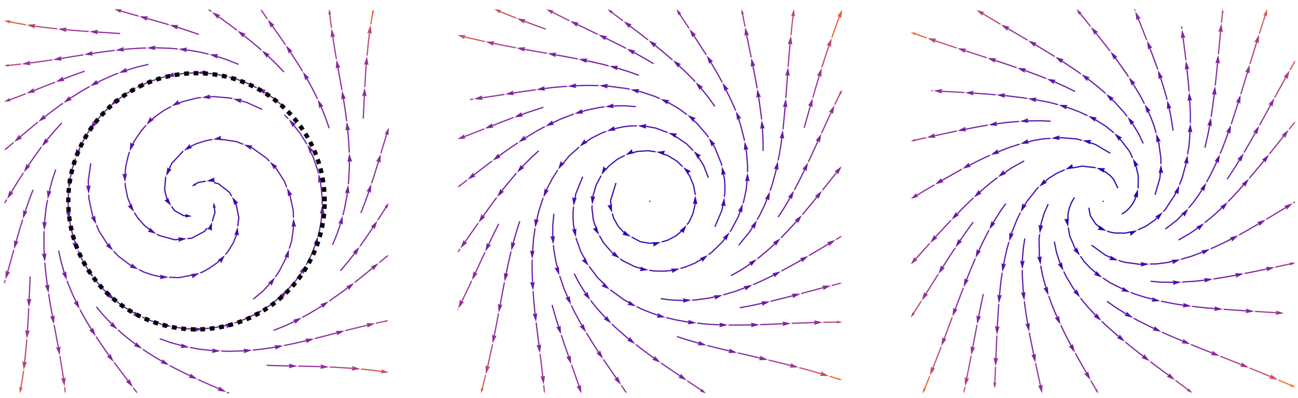
so eigenvalues are given by

$$\lambda_{\pm} = \mu \pm i\omega.$$

For $\omega > 0$ (we'll only consider this case; the case $\omega < 0$ is symmetrical, as we are dealing with a conjugate pair of eigenvalues), as μ is varied and crosses zero, we see that the stability of the origin changes but the number of fixed points remains constant. Namely, when $\mu > 0$ the origin is unstable, and vice versa.

Returning to polar, we see that $\dot{r} = 0$ only if either $r = 0$ or $r = \sqrt{\frac{\mu}{a}}$. $\dot{\theta}$ is constant, so this implies that there is a circular orbit of radius $r = \sqrt{\mu/a}$ and period $2\pi/\omega$, whenever μ and a have the same sign.

For $a < 0$, this periodic orbit is unstable and the origin must be stable, so this is called a *subcritical Hopf*; for $a > 0$, the orbit is stable, the origin is unstable and we have a *supercritical Hopf*; finally, for $a = 0$, the origin is stable for $\mu < 0$ and vice versa, and when $\mu = 0$, everything is periodic (the phase space consists only of concentric circles).



A subcritical Hopf bifurcation

↪**Theorem 6.3** (Conditions for a Hopf Bifurcation): let $\dot{x} = f(x, y, \mu)$ and $\dot{y} = g(x, y, \mu)$ with $f(0, 0, \mu) = 0 = g(0, 0, \mu)$ for all μ, and Jacobian at $(0, 0)$ given by $\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$, for some $\omega \neq 0$. Then, if $f_{\mu x} + g_{\mu y} \neq 0$ and $a \neq 0$, where

$$a := \frac{1}{16}\left(f_{xxx} + g_{xxy} + f_{xyy} + g_{yyy}\right) + \frac{1}{16\omega}\left(f_{xy}\left(f_{xx} + g_{yy}\right) - g_{xy}\left(g_{xx} + g_{yy}\right) - f_{xx}g_{xx} - f_{yy}g_{yy}\right),$$

then a curve of periodic orbits bifurcates from the origin into $\mu < 0$ if $a\left(f_{x\mu} + g_{y\mu}\right) > 0$ or into $\mu < 0$ if $a\left(f_{x\mu} + g_{y\mu}\right) < 0$.

- The steady state at the origin is stable for $\mu > 0$ and unstable for $\mu < 0$ if $f_{\mu x} + g_{\mu y} < 0$, and the opposite for $f_{\mu x} + g_{\mu y} > 0$.
- The periodic orbit is stable/unstable if the origin is unstable/stable.
- The amplitude of the periodic orbit grows according to $|\mu|^{1/2}$, and need not in general be circular. The period converges to $2\frac{\pi}{|\omega|}$ as $|\mu| \to 0$.

The bifurcation is called supercritical if the periodic orbit is stable, and subcritical if the periodic orbit is unstable.

§6.4 Takens-Bogdonov Bifurcation

In the previous examples, we've dealt with bifurcations that have a single parameter being varied. While this can lead to a wide range of dynamic changes in the system, there are certain dynamics and in particular certain bifurcations that are only possible if we allow multiple parameters to vary.

Consider the system

$$\dot{x} = y, \qquad \dot{y} = \mu_1 + \mu_2 y + x^2 + xy.$$

This has

- 0 steady states if $\mu_1 > 0$;
- 1 steady state at the origin $(0, 0)$ if $\mu_1 = 0$;
- 2 steady states at $\left(\pm\sqrt{-\mu_1}, 0\right)$ if $\mu_1 < 0$,

which implies a fold bifurcation occurs when μ_1 changes sign.

We consider now different cases of μ_1.

[$\mu_1 > 0$]: here, we find

- $y > 0 \Rightarrow \dot{x} > 0$;
- $y < 0 \Rightarrow \dot{x} < 0$;
- $y = 0 \Rightarrow \dot{x} = 0$ and $\dot{y} > 0$, from which we may conclude there are not only no steady states, but also no periodic orbits.

In short, generally boring behaviour.

[$\mu_1 < 0$] We find

$$J_f\left(\pm\sqrt{-\mu_1}, 0\right) = \begin{pmatrix} 0 & 1 \\ \pm 2\sqrt{-\mu_1} & \mu_2 \pm \sqrt{-\mu_1} \end{pmatrix},$$

which has eigenvalues satisfying

$$-\lambda(\mu_2 + x - \lambda) - 2x = 0$$
$$\Rightarrow \lambda^2 - (\mu_2 + x)\lambda - 2x = 0.$$

For a steady state bifurcation, we'd need $\mu_2 = 0$ and thus $\mu_1 = 0$, so not possible; thus no other bifurcations can occur then the one we already have. Hence, we have

$$\lambda_\pm = \frac{(\mu_2 + x) \pm \sqrt{(\mu_2 + x)^2 + 8x}}{2}.$$

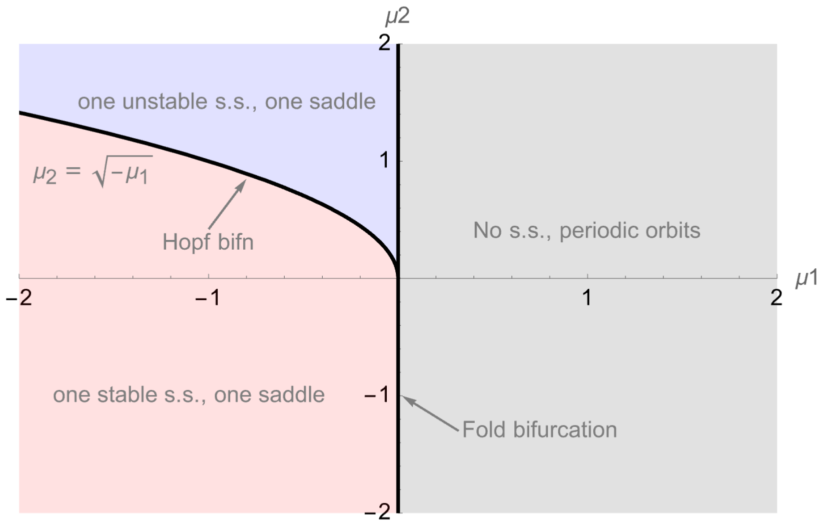
For the point $x = \left(\sqrt{-\mu_1}, 0\right)$, $x > 0$ so $\lambda_+ > 0 > \lambda_-$, so we the steady state is a saddle point for every μ_2.

For $x = \left(-\sqrt{-\mu_1}, 0\right)$, if $\mu_2 = \sqrt{-\mu_1}$ then $\mu_2 + x = 0$, so $\lambda_\pm = \pm i\sqrt{2}(-\mu_1)^{1/4}$, so we have a pair of complex conjugate eigenvalues.

For $\mu_2 \approx \sqrt{-\mu_1}$, then $(\mu_2 + x)^2 + 8x \approx 8x < 0$, so again a pair of complex conjugate eigenvalues, with

$$\mathrm{Re}(\lambda_\pm) \approx \frac{\mu_2 + x}{2} = \frac{1}{2}\left(\sqrt{\mu_2} - \sqrt{-\mu_1}\right),$$

which is negative if $\mu_2 < \sqrt{-\mu_1}$ (hence stable) and positive if $\mu_2 > \sqrt{-\mu_1}$ (hence unstable). So in particular, $\mathrm{Re}(\lambda_\pm)$ changes sign when $\mu_2 = \sqrt{-\mu_1}$, at which point we see we have a Hopf bifurcation.



§7 Maps

Given a function $f : U \to U$ for some subset $U \subseteq \mathbb{R}^n$, a map is defined by the iteration $x_{n+1} = f(x_n)$ for $n \in \mathbb{N}$, with some initial condition $x_0 \in U$.

§7.1 Linearization

A fixed point of a map satisfies $f(x^*) = x^*$; so in particular, $f^n(x^*) = x^*$ for every $n \geq 1$. Let $y_n = x_n - x^*$ so

$$
\begin{aligned}
y_{n+1} &= f(x^* + y_n) - x^* \\
&= \left[f(x^*) + J_f(x^*)y_n + \mathcal{O}(2)\right] - x^* \\
&= J_f(x^*)y_n + \mathcal{O}(2),
\end{aligned}
$$

hence, the linearization of the map is given by $y_{n+1} = J_f(x^*)y_n$. Suppose (λ, v) an eigenpair of $J_f(x^*)$. Then, let $y_0 = v$; then,

$$
y_1 = J_f v = \lambda v \Rightarrow y_2 = \lambda^2 v \Rightarrow \cdots \Rightarrow y_n = \lambda^n v.
$$

In particular, then, we see that if $|\lambda| > 1$, $|y_n| \to \infty$, while if $|\lambda| < 1$, $|y_n| \to 0$. With this in mind, then, we say that x^* *unstable* if $|\lambda| > 1$ for any eigenvalue λ of the Jacobian at x^*, and *stable* if $|\lambda| < 1$ for every eigenvalue λ. Remark this is quite different than the stability requirements for a linear ODE, which looks at whether at an eigenvalue is positive or negative. In particular, this analysis works for both real and complex eigenvalues, where we take in the latter case $|\cdot|$ to be the modulus of the eigenvalue.

↪**Definition 7.1** (Hyperbolic): A fixed point x^* of a map is *hyperbolic* if $|\lambda| \neq 1$ for every eigenvalue λ of the Jacobian $J_f(x^*)$.

We have then analogous definitions of the linear stable, unstable, and centre manifolds for maps as in the ODE case, as well as:

$$
W^s(x^*) := \{x \mid f^n(x) \to x^*\}, \qquad W^u(x^*) := \{x \mid \Gamma^-(x) \exists, f^n(x) \to x^*\}.
$$

↪**Theorem 7.1** (Stable Manifold): Suppose $x^* = 0$ a hyperbolic fixed point for the map $x_{n+1} = f(x_n)$ where f a diffeomorphism. Then, $y_{n+1} = J_f(0)y_n$ has stable, unstable manifolds E^s, E^u which are tangential to the stable, unstable nonlinear manifolds W^s, W^u of $x^* = 0$.

§7.2 Bifurcations

↪**Theorem 7.2** (Centre Manifold Theorem):

§7.3 Period Doubling Bifurcation

§7.4 Naimark-Sacker Bifurcation

§7.5 Sharkovski's Theorem

§7.6 Shilnikov/Homoclinic Bifurcation