MATH325 - ODEs
Summary of Results

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Complete notes

1 Notation and Terminology

Definition 1 (Order)**.** The order of a differential equation is the order of the highest derivative in the equation.

Definition 2 (Autonomous/Nonautonomous, Linear/Nonlinear, Homogeneous/Nonhomogeneous, Constant/Variable)**.**

$$y^{(n)}(x) = \underbrace{f(y, y', \dots, y^{(n-1)})}_{\text{no } x} \quad - \quad \text{autonomous}$$

$$y^{(n)}(x) = f(\mathbf{x}, y, y', \dots, y^{(n-1)}) \quad - \quad \text{nonautonomous}$$

$$
\begin{aligned}
\circledast := \sum_{i=0}^{n} a_i(t) y^i(t) = g(t) \quad &- \quad \text{linear} \\
\cdots \text{ otherwise } \cdots \quad &- \quad \text{nonlinear} \\
\circledast \text{ with } g(t) \equiv 0 \quad &- \quad \text{homogeneous} \\
\circledast \text{ with } g(t) \not\equiv 0 \quad &- \quad \text{nonhomogeneous} \\
\circledast \text{ with } a_i^{\prime}s \text{ constant} \quad &- \quad \text{constant} \\
\circledast \text{ with } a_i^{\prime}s \text{ variable} \quad &- \quad \text{variable}
\end{aligned}
$$

Equivalently, linear equations can be defined by having their solution space defining a vector space.

Definition 3 (Solution)**.** A function $y : I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ is said to be a solution to an nth order ODE if it it is n-times differentiable on I and satisfies the ODE on that interval.

Definition 4 (Interval of Validity)**.** The interval of validity of a solution to an ODE $I \subseteq \mathbb{R}$ is the largest interval for which $y(t)$ solves the ODE.

We will use $L[y](x)$ linear operator as shorthand for differential equations.

2 First Order

Remark that this is the only section where we will truly concern ourselves with both linear *and* nonlinear equations.

Proposition 1 (Separable)**.** *An ODE of the form*

$$y' = P(t)Q(y)$$

is said to be separable, and has general solution by integrating

$$\int \frac{1}{Q} \, \mathrm{d}y = \int P(t) \, \mathrm{d}t \, .$$

Proposition 2 (Linear First Order). *An ODE of the form*

$$a_1(t)y'(t) + a_0(t)y(t) = g(t) \rightsquigarrow y'(t) + p(t)y(t) = q(t)$$

is called linear, and with "integrating factor" $\mu(t) := e^{\int p(t)\mathrm{d}t}$ *can be written*

$$\mathrm{d}t\,(\mu(t)y(T)) = \mu(t)q(t),$$

with general solution found by integrating both sides and solving for y.

Proposition 3 (Exact). *An ODE of the form*

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0$$

is said to be exact, if $M_y = N_x$. *If so, it has general solution* $F(x,y) = C$ *where* $F_x = M, F_y = N$, *and* C *an arbitrary constant.*

Proposition 4 ("Exactable"). *For equations "almost" exact, one may find a* $\mu = \mu(x,y)$ *such that*

$$\frac{\partial}{\partial x}(\mu M) = \frac{\partial}{\partial y}(\mu N),$$

in which case the new ODE $\mu M\,\mathrm{d}x + \mu N\,\mathrm{d}y = 0$ *is now exact.*

Remark 1. Simplifying by assuming $\mu_x = 0$ or $\mu_y = 0$ can help immensely.

Proposition 5 (Bernoulli). *An ODE of the form*

$$y' + f(x)y + g(x)y^n = 0$$

are called Bernoulli, and can be transformed into a linear equation by the substitution $u = y^{1-n}$.

Proposition 6 (Other Substitutions).

- *Homogeneous equations can be transformed into separable equations by substitution* $u := \frac{y}{x}$

- *Equations of the form $y' = F(ay + bx + c)$ can be solved via $u := ay + bx + c$.*

Remark 2. Other substitution methods exist, of course; these three are the more common.

Theorem 1 (★ Existence, Uniqueness)**.** *If $f(t, y), f_y(t, y)$ continuous in t, y on a rectangle $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$, then $\exists h \in (0, a]$ such that the IVP*

$$y' = f(t, y), \quad y(t_0) = y_0$$

has a unique solution defined for $t \in [t_0 - h, t_0 + h]$, with $y(t) \in [y_0 - b, y_0 + b] \, \forall \, t \in [t_0 - h, t_0 + h]$.

Remark 3. While the details of the proof are not too vital (?), the requirements for the theorem to hold (namely, continuity) are. In particular, recall that in the proof, we take $h < \min\{a, \frac{1}{L}, \frac{b}{M}\}$, where a, b defined by the box, L the Lipschitz constant of f, and M the upper bound of f on the box.

Definition 5 (Picard Iteration)**.** For the IVP $y' = f(t, y), y(0) = y_0$, define a sequence $y_n(t)$ as follows; $y_0(t) := y_0 \, \forall \, t$, and

$$y_{n+1}(t) := y(t_0) + \int_{t_0}^{t} f(s, y_n(s)) \, \mathrm{d}s \,, \quad \forall \, n \geqslant 1.$$

Remark 4. Denoting $T : C(I) \to C(I), y_n \mapsto y(t_0) + \int_{t_0}^{t} f(s, y_n(s)) \, \mathrm{d}s$, then y solves the IVP iff $Ty = y$. Indeed, to see the motivation for Picard iteration directly, integrate both sides of the IVP.

3 Second Order

Equations in this section will be of the general form $y'' = f(t, y, y')$.

Proposition 7 (Special Cases)**.**
- *If $y'' = f(t, y')$, letting $u = y'$ yields a first-order $u' = f(t, u)$, which can be solved with techniques from the previous section, then the solution u can be integrated to find y.*

- *If $y'' = f(y, y')$, letting $u = y'$ yields $u' = f(y, u)$; by the chain rule $\frac{du}{dt} = u\frac{du}{dy}$, so we have again a first order ODE, this time with $u = u(y)$.*

Proposition 8 (Superposition). *If $y_1, \ldots, y_n$ solve $L[y](t) = 0$ on some interval I, so does $\sum_{i=1}^{n} a_i y_i(t)$ for arbitrary constants a_i.*

Proposition 9 (⋆ Reduction of Order). *Given a solution $y_1(t)$ to $a(t)y'' + b(t)y' + c(t)y = 0$, then taking $y(t) = u(t)y_1(t)$, we can then reduce the equation to a first-order ODE of the form $0 = [ay_1]v' + [2ay_1' + by_1]v$, where $v = u'$, which we can then solve for v, hence u, then y a new solution.*

Proposition 10 (⋆ Constant Coefficient). *For an equation of the form*

$$ay'' + by' + cy = 0,$$

where a, b, c constants, we have the corresponding characteristic/auxiliary equation

$$ar^2 + br + c = 0,$$

with roots r_1, r_2, and solutions

- $r_1 \neq r_2 \in \mathbb{R} \implies y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$
- $r := r_1 = r_2 \implies y_1 = te^{rt}, y_2 = e^{rt}$.
- $\alpha + \beta i := r_1 = \overline{r_2} \in \mathbb{C} \implies y_1 = e^{\alpha t}\cos(\beta t), y_2 = e^{\alpha t}\sin(\beta t)$.

Definition 6 (Particular Solution). A solution y_p of an ODE is said to be a a particular solution if it it solves $L[y] = g(t) \neq 0$.

Proposition 11 (Undetermined Coefficients). *For $L[y] = ay'' + by' + cy = g(t)$, then if $g(t)$ of the following form (left), guessing y_p (right) will yield a particular solution after solving for the constants (by plugging into $L[y]$): where s the multiplicity of the root $\alpha + i\beta$ if it is a root of the auxiliary equation, and 0 otherwise.*

$g(x)$ (given)	$y_{p(x)}$ (guess)
$p(x)$	$x^s(A_n x^n + \cdots + A_1 x + A_0)$
$e^{\alpha x}$	$x^s A e^{\alpha x}$
$p(x)e^{\alpha x}$	$x^s(A_n x^n + \cdots + A_1 x + A_0)e^{\alpha x}$
$p(x)e^{\alpha x}\cos\beta x + q(x)e^{\alpha x}\sin\beta x$	$x^s e^{\alpha x}\cos(\beta x)\sum_{i=0}^n A_i x^i +$ $x^s e^{\alpha x}\sin(\beta x)\sum_{j=0}^n B_j x^j.$

Remark 5. Only works for constant coefficient!

Proposition 12 (★ Variation of Parameters)**.** *Let $L[y](x) = a(x)y'' + b(x)y' + c(x)y = g(x)$. Given a fundamental set of solutions y_1, y_2, then guessing a particular solution $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$, then after appropriate mathematical silliness, u_1, u_2 satisfy*

$$u_1' = \frac{-y_2(x)\frac{g(x)}{a(x)}}{W(y_1, y_2)(x)}, \quad u_2' = \frac{y_1(x)\frac{g(x)}{a(x)}}{W(y_1, y_2)(x)},$$

where $W(y_1, y_2) = y_1 y_2' - y_2 y_1'$, such that y_p solves the ODE.

Proposition 13. *Both of these previous methods can be extended to higher-order linear ODEs, with variation of parameters being rather hellish. Remark that variation of parameters works for non-constant coefficient linear equations.*

4 Nth Order

We consider nth order ODEs of the form $L[y] = y^{(n)} + \sum_i^n p_i(x)y^{(n-1)}(x) = g(x)$; $L[y]$ refers to this form unless otherwise noted. This section will mostly be the heaviest theory-wise, and will also cover results applicable, naturally, to 2nd order ODEs.

Proposition 14 (Uniquess and Existence)**.** *Let $I \subseteq \mathbb{R}$, $x_0 \in I$ and let $p_i(x)$, $i = 1, \ldots, n$ and $g(x)$ be continuous on I. Then, the IVP*

$$L[y](x) = g(x) \quad y^{(j)}(x_0) = \alpha_{j+1}, j = 0, \ldots, n-1$$

has at most one solution $y(x)$ defined on I.

Definition 7 (Fundamental Set of Solutions)**.** A set of functions $\{y_i : L[y_i] = 0, i = 1, \ldots, n\}$ on some interval I is called a fundamental set of solutions if $y_1, \ldots, y_n$ are linearly independent on I.

Remark 6. I may change such that y_i are no longer independent!

Definition 8 (Wronskian)**.** Put

$$W(y_1, \ldots, y_n)(x) := \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & \cdots & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}.$$

Proposition 15. *If $W(y_1, \ldots, y_n)(x_0) \neq 0$ for some $x_0 \in I$ then $y_1, \ldots, y_n$ are linearly independent on I. If $y_1, \ldots, y_n$ are linearly dependent on I, then $W(y_1, \ldots, y_n)(x) = 0 \, \forall \, x \in I$.*

Remark 7. Very important: this statement does NOT hold iff; more precisely, $W(y_1, \ldots, y_n)(x) = 0 \, \forall \, x \in I$ does NOT imply $y_1, \ldots, y_n$ linearly dependent on I; consider for instance

$$y_1 = x^2, \quad y_2 = \begin{cases} x^2 & x \geqslant 0 \\ -x^2 & x \leqslant 0 \end{cases},$$

which has Wronskian 0 everywhere but are clearly not linearly independent on I.

In order to "have the converse hold", we must have that the $y_1, \ldots, y_n$ solve a particular ODE (to make precise to follow).

Theorem 2 (★ Abel's)**.** *Let $y_1, \ldots, y_n$ solve $L[y] = 0$ where $p_j(x)$'s continuous, all on some I. Then*

$$W'(x) + p_1(x)W(x) = 0 \, \forall \, x \in I.$$

Moreover, this being a linear equation, we have that

$$W(x) = Ce^{-\int p_1(x)\mathrm{d}x}.$$

As a consequence, either

- $C = 0$ *so* $W \equiv 0$ *and* $y_1, \dots, y_n$ *linearly dependent on* I*;*
- $C \neq 0$ *so* $W \neq 0 \, \forall \, x \in I$ *and* $y_1, \dots, y_n$ *linearly independent on I and so form a fundamental set of solutions.*

Remark 8. Remark the continuity of the p_j's- this is crucial. One can construct counter examples in the case that p_j's not continuous on I.

The second ("as a consequence") part of the theorem follows directly from the exponential function being a strictly positive function. Verbally, either the Wronskian is nowhere 0, or, if 0 at a single point, is identically 0. Again, to emphasize, this holds in this case as we are now working with a set of solutions. More precisely:

Corollary 1. *With the same assumptions as in Abel's Theorem, TFAE:*

1. $y_1, \dots, y_n$ *form a fundamental set of solutions on* I*;*
2. $y_1, \dots, y_n$ *are linearly independent on* I*;*
3. $W(y_1, \dots, y_n)(x_0) \neq 0$ ***for some*** $x_0 \in I$*;*
4. $W(y_1, \dots, y_n)(x) \neq 0$ ***for all*** $x \in I$*.*

Remark 9. The converse, naturally, holds as well ($W = 0$ for some point iff $W \equiv 0$).

Theorem 3. *If* $y_1, \dots, y_n$ *a fundamental set of solutions for* $L[y] = 0$ *on* I *with continuous* $p_j(x)$ *on* I*, then the IVP* $L[y] = 0, y(x_0) = \alpha_1, \dots, y^{(n-1)}(x_0) = \alpha_n$ *has a unique solution of the form* $\sum_{i=1}^{n} c_j y_j(x)$ *for unique constants* c_j*.*

Similarly, for $L[y] = g$ *with the same IVP conditions, any solution can be written in the form* $y_p(x) + \sum_{j=1}^{n} c_j y_j(x)$ *where* $L[y_p] = g$ *and* c_j *unique constants.*

Sketch. To show the form being unique, construct a system of n linear equations in the n unknowns $c_1, \dots, c_n$ in terms of the equations and α_i's. In matrix form, you should find the matrix that the Wonskian is the determinant of appear, and since the Wronskian nonzero by assumption of a fundamental set of solutions, you can invert, which simultaneously gives existence and uniqueness as per uniqueness of inverses. $\square$

Proposition 16 (Higher-Order Variation of Parameters)**.** *Given $y_1, \ldots, y_n$ a fundamental set of solutions to $L[y] = 0$, let $W_i(x)$ be the determinant of the matrix obtained by replacing the ith column of W with $\begin{pmatrix} 0 \\ \vdots \\ g \end{pmatrix}$. Then, taking $u_i := \int_{x_0}^{x} \frac{W_i(s)}{W(s)}\,\mathrm{d}s$, then*

$$y_p = \sum_{i=1}^{n} u_i(x) y_i(x)$$

a particular solution to $L[y] = g$.

5 Series

We again only consider linear equations, but now have the tools to work with non-constant coefficient equations more generally. As a motivation, series solutions can be thought of as approximating ugly solutions arbitrarily well via polynomials (which hopefully converge?).

Proposition 17. *Let $f(x) := \sum_{n=0}^{\infty} a_n (x - x_0)^n$, $g(x) := \sum_{n=0}^{\infty} b_n (x - x_0)^n$ and ρ_f, ρ_g the radii of converge of f, g resp. The radius of converge of $f \pm g$ and $f \cdot g$ is at least as large as* $\min\{\rho_f, \rho_g\}$.

Remark 10. We won't worry about dividing power series, but this can result in a smaller radius of convergence than either ρ_f, ρ_g.

Proposition 18 (Important Power Series to Remember)**.**

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

These each have infinite radius of convergence.

Any polymomial $f(x) = a_0 + a_1 x + \cdots + a_N x^N$ has power series $\sum_{n=0}^{\infty} \tilde{a}_n x^n$, where $\tilde{a}_n := \begin{cases} a_n & n \leqslant N \\ 0 & n > N \end{cases}$, and also has infinite radius of convergence.

Definition 9 (Analytic)**.** We say $P : I \to \mathbb{R}$ analytic at $x_0 \in I$ if there exist a power series representation of P centered at x_0 with nonzero radius of convergence.

Proposition 19. *If $P(x), Q(x)$ polynomials, $\frac{Q(x)}{P(x)}$ analytic at x_0 if $P(x_0) \neq 0$; when analytic, the radius of convergence from x_0 is the distance from x_0 to the nearest zero of $P(x)$ in the complex plane.*

Definition 10 (Ordinary, Singular)**.** Let $L[y] = P(x)y'' + Q(x)y' + R(x)y$ and $p(x) := \frac{Q}{P}, q(x) := \frac{R}{P}$. We say x_0 an ordinary point of $L[y] = 0$ if both p, q are analytic at x_0. Else, we call x_0 a singular point. Moreover, if P, Q, R polynomials, then if $P(x_0) \neq 0$, x_0 an ordinary point, and if $P(x_0) = 0$, x_0 a singular point.

For singular points, if

$$(x - x_0)p(x), \quad (x - x_0)^2 q(x)$$

are both analytic at x_0, then we say x_0 a regular singular point, and irregular if either is not analytic at x_0. In particular, if P, Q, R polynomials, x_0 a regular singular point iff x_0 a singular point and the limits of both of these expressions as $x \to x_0$ are finite.

Proposition 20 (★ General Method for Ordinary Points, Homogeneous)**.** *Let $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$. Plugging into $L[y] = 0$, one can find a recursive definition for $a_n, n \geqslant 2$, with a_1, a_0 arbitrary (determined by IC's), which can be written as $y(x) = a_0 y_1(x) + a_1 y_2(x)$ where y_1, y_2 analytic at x_0, have radius of convergence at least as large as the minimum of p, q, form a fundamental set of solutions, and have Wronksian* 1.

Remark 11. Series are best learned by doing examples.

In the case where p, q are not polynomials, we have a bit more work to do; you need to represent both as power series, then multiply the power series together...

Proposition 21 (General Method, Nonhomogeneous). *For $L[y] = g(x), g(x)$ analytic, a remarkably similar process follows, by representing $g(x)$ as a power series and again equation like powers of x. In this case, we'll find a general solution of the form*

$$a_0 y_1 + a_1 y_2 + y_p,$$

where y_1, y_2, y_p analytic (usually we end up with power series in solutions) and y_p has no reliance on a_0, a_1 and satisfies $L[y_p] = g$.

Theorem 4 (Regular Singular Points - Frobenius's Method). *If x_0 a regular singular point of $L[y] = 0$, seek a solution of the form $y(x) = |x - x_0|^r \sum_{n=0}^{\infty} a_n (x - x_0)^n$ (it suffices to assume $x - x_0 > 0$ for sake of removing the absolute value bars). This results in the indicial equation*

$$F(r) = r(r-1) + rp_0 + r_0 = 0,$$

where $p_0 = \lim_{x \to x_0}(x - x_0)p(x), q_0 = \lim_{x \to x_0}(x - x_0)^2 q(x)$. Let $r_1 \geqslant r_2$ be the two real roots of F (we won't consider the complex case). Then, we have one solution of the form

$$y_1 = |x - x_0|^{r_1} \sum_{n=0}^{\infty} a_n(r_1)(x - x_0)^n,$$

where $a_1 = 1$, and a second of the form

- $(r_1 - r_2 \neq 0$ *and* $r_1 - r_2 \notin \mathbb{Z}), y_2 = |x - x_0|^{r_2} \sum_{n=0}^{\infty} a_n(r_2)(x - x_0)^n$
- $(r_1 = r_2), y_2 = y_1(x) \ln |x - x_0| + |x - x_0|^{r_1} \sum_{n=1}^{\infty} b_n (x - x_0)^n$, *where b_n TBD*
- $(r_1 - r_2 = N \in \mathbb{N}), y_2 = a y_1(x) \ln |x - x_0| + |x - x_0|^{r_2} \sum_{n=0}^{\infty} c_n (x - x_0)^n$, *where $a = \lim_{r \to r_2}(r - r_2) a_N(r)$ and c_n some series depending on $a_n(r_2)$.*

Remark 12. You will probably only have to deal with the first and maybe second cases.

For the second case, "normalizing" (ie, making sure the y'' term has an x^2) the ODE is vital.

Remark 13. We won't concern ourselves with irregular singular points.

6 Laplace Transformations

Remark that most equations treated in this section can be treated with previous techniques; only equations with constant coefficients are treated. Note too that most of the theorems/Laplace identities we state are proven via (repeated) integration by parts. Actual questions are solved via applied partial fraction theory.

Definition 11 (Laplace Transform)**.** For $f : [0, \infty) \to \mathbb{R}$, we denote

$$F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty e^{-st} f(t)\,\mathrm{d}t\,.$$

Remark 14. Practically, you won't have to apply the definition directly too often and will be given a table of common transforms. It can be helpful for certain proofs, of course.

Definition 12 (Exponential Order)**.** A function $f(t)$ is said to be of exponential order a if $\exists a, K, T$-constants such that $|f(t)| \leqslant Ke^{at}\,\forall\, t \geqslant T$.

Theorem 5. *If f piecewise continuous on $[0, \infty)$ and has exponential order a, then $\mathcal{L}\{f(t)\}$ exists for $s > a$.*

Sketch. Subdivide the interval of integration so that you are integrating over time larger thatn T, and apply the exponential order condition. □

Proposition 22. *$\mathcal{L}\{\dots\}$ linear.*

Theorem 6 (★ First Translation Theorem)**.** $\mathcal{L}\{e^{kt}f(t)\} = F(s-k) \equiv \mathcal{L}\{f(t)\}_{s\to s-k}$

Theorem 7 (★)**.** *If $f, \dots, f^{(n-1)}$ continuous on $[0, \infty)$ and $f^{(n)}$ piecewise continuous on $[0, \infty)$ and all are of exponential order a, then for $s > a$*

$$\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - \sum_{k=0}^{n-1} s^{n-1-k} f^k(0).$$

Remark 15. This is the crucial theorem to apply Laplace transforms to solving IVPs. We remark the $n = 1, 2$ cases as these will be the most often used:

$$\mathcal{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$$
$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

Corollary 2. *Given $L[y] = \sum_{k=0}^{n} a_k y^{(k)} = f(t)$, $y(0) = \alpha_1, \ldots, y^{(n-1)}(0) = \alpha_n$, we have*

$$Y(s) = \frac{F(s)}{P(s)} + \frac{Q(s)}{P(s)} = G(s) + \frac{Q(s)}{P(s)},$$

where $F(s) = \mathcal{L}\{f(t)\}$, $P(s) = a_n s^n + \cdots + a_1 s + a_0$ the characteristic equation, and $Q(s)$ some polynomial in s of degree leq $n - 1$.

Remark 16. $\deg(P) > \deg(Q)$ gives us that we can rewrite this term in terms of simpler expressions using partial fractions to find the inverse Laplace transform.

Definition 13 (Unit Step Function)**.** Put $\mathcal{U}(t-a) := \begin{cases} 0 & t < a \\ 1 & t \geqslant a \end{cases}.$

Theorem 8 (★ Second Translation Theorem)**.** *For $a > 0$, $\mathcal{L}\{\mathcal{U}(t-a)f(t-a)\} = e^{-as}F(s)$.*

Corollary 3. $\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$.

Proposition 23. $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{L}\{f(t)\}$.

Definition 14 (Convolution)**.** Put $(f * g)(t) := \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau$.

Theorem 9 (Convolution Theorem)**.** *If f, g piecewise continuous on $[0, \infty)$ and of exponential order,*

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}.$$

In particular,

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f * g.$$

Definition 15 (Dirac Delta)**.** Let $\delta(t - t_0)$ be such that $\int_{-\infty}^{\infty} f(t)\delta(t - t_0)\,\mathrm{d}t = f(t_0)$. In particular,

$$\int_0^t \delta(s - t_0)\,\mathrm{d}s = \begin{cases} 0 & t < t_0 \\ 1 & t > t_0 \end{cases}.$$

Remark 17. It is possible to be more rigorous in our definition of δ, but beyond this scope of this course.

Theorem 10. *For* $t_0 > 0$, $\mathcal{L}\{\delta(t - t_0)\} = e^{-st_0}$.

Corollary 4. $\mathcal{L}\{\delta(t)\} = 1$.

Definition 16 (Green's Function)**.** $g(t)$ such that $L[g(t)] = \delta(t)$ with IC $g(0) = g'(0) = \cdots = g^{(n-1)}(0)$.

Theorem 11. $\mathcal{L}\{g(t)\} = \frac{1}{P(s)}$.

Theorem 12. *Let f be periodic with period T and piecewise continuous on* $[0, \infty)$. *Then*

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\,\mathrm{d}t\,.$$